ALLEN CAREER INSTITUTE KOTA (RAJASTHAN) Path to Success

Contents

KINEMATICS (MOTION ALONG A STRAIGHT LINE AND MOTION IN A PLANE)

E

NEET SYLLABUS

Frame of reference, Motion is a straight line ; position -time graph, speed and velocity. Uniform and nonuniform motion, average speed and instantaneous velocity. Uniformly accelerated motion, velocity-time and position-time graphs, for uniformly accelerated motion (graphical treatment).

Motion in a plane. Case of uniform velocity and uniform acceleration -projectile motion.

KINEMATICS

KINEMATICS

Study of motion of objects without taking into account the factors which cause the motion (i.e. nature of force).

1. FRAME OF REFERENCE

Motion of a body can be observed only if it changes its position with respect to some other body. Therefore, for motion to be observed there must be a body, which is changing its position with respect to another body and a person who is observing motion. The person observing motion is known as observer. The observer for the purpose of investigation must have its own clock to measure time and a point in the space attached with the other body as origin and a set of coordinate axes. These two things (the time measured by the clock and the coordinate system) are collectively known as reference frame.

In this way, motion of the moving body is expressed in terms of its position coordinates changing with time.

2. MOTION & REST

If a body changes its position with time, it is said to be moving otherwise it is at rest. Motion/rest is always relative to the observer.

Motion/rest is a combined property of the object under study and the observer. There is no meaning of rest or motion without the observer or frame of reference.

- To locate the position of a particle we need a reference frame. A commonly used reference frame is cartesian coordinate system or x-y-z coordinate system.

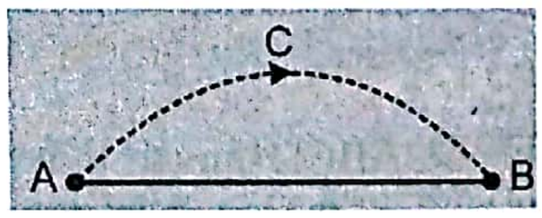
  The coordinates (x, y, z) of the particle specify the position of the particle with respect to origin of that frame. If all the three coordinates of the particle remain unchanged as time passes it means the particle is at rest w.r.t. this frame.
- If only one coordinate changes with time, motion is one dimensional (1 - D) or straight line motion.

  If only two coordinates change with time, motion is two dimensional (2 - D) or motion in a plane. If all three coordinates change with time, motion is three dimensional (3 - D) or motion in space.
- The reference frame is chosen according to problem.
- If frame is not mentioned, then ground is taken as reference frame.

3. DISTANCE & DISPLACEMENT

Distance

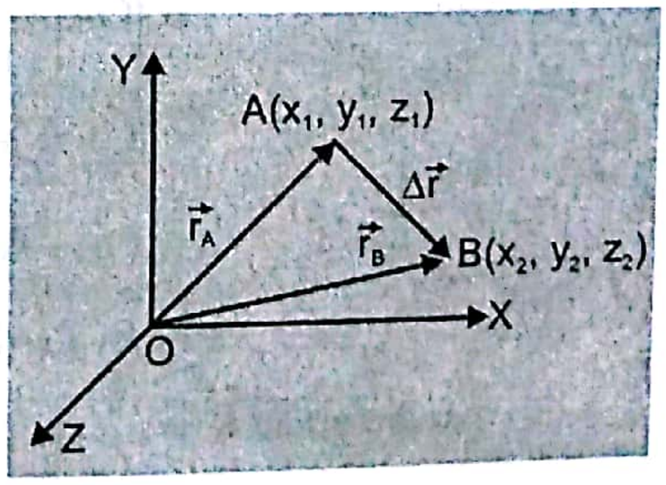
Distance is total length of path covered by the particle, in definite time interval.



Let a body moves from A to B via C. The length of path ACB is called the distance travelled by the body.

But overall, body is displaced from A to B. A vector from A to B, i.e. $\vec{AB}$ is its displacement vector or displacement that is the minimum distance and directed from initial position to final position.

Displacement in terms of position vector



Let a body be displaced from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ then its displacement is given by vector AB.

From ΔOAB $\quad \vec{r}_A + \Delta\vec{r} = \vec{r}_B \quad$ or $\quad \Delta\vec{r} = \vec{r}_B - \vec{r}_A$

$\because \quad \vec{r}_B = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{r}_A = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\therefore \quad \Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \quad$ or $\quad \Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$

GOLDEN KEY POINTS

- Distance is a scalar while displacement is a vector.
- Distance depends on path while displacement is independent of path but depends only on final and initial positions.
- For a moving body, distance cannot have zero or negative values but displacement may be positive, negative or zero.
- Infinite distances are possible between two fixed points because infinite paths are possible between two fixed points.
- Only single value of displacement is possible between two fixed points.
- If motion is in straight line without change in direction then

  distance = | displacement | = magnitude of displacement.
- Magnitude of displacement may be equal or less than distance but never greater than distance.

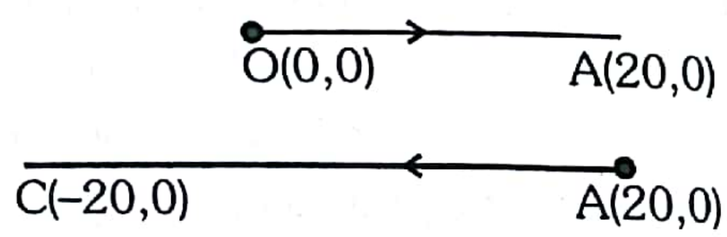
  i.e., distance ≥ | displacement |

Illustrations

Illustration 1.

A particle starts from the origin, goes along the X-axis upto the point (20m, 0) and then returns along the same line to the point (–20m, 0). Find the distance and displacement of the particle during the trip.

Solution



Distance = | OA | + | AC |

= 20 + 40 = 60m

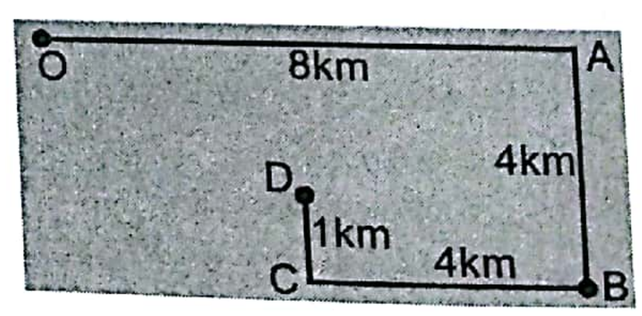
Displacement = OA + AC

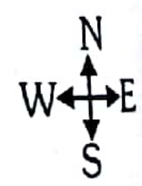
$= 20\hat{i} + (-40\hat{i}) = (-20\hat{i})$m

Illustration 2.

A car moves from O to D along the path OABCD shown in fig.

What is distance travelled and its net displacement ?





Solution

Distance $= |\vec{OA}| + |\vec{AB}| + |\vec{BC}| + |\vec{CD}|$

$= 8 + 4 + 4 + 1 = 17$ km

Displacement $= \vec{OA} + \vec{AB} + \vec{BC} + \vec{CD}$

$= 8\hat{i} + (-4\hat{j}) + (-4\hat{i}) + \hat{j} = 4\hat{i} - 3\hat{j}$

$\Rightarrow$ | displacement | $= \sqrt{(4)^2 + (3)^2} = 5$

So, Displacement = 5 km, 37° S of E

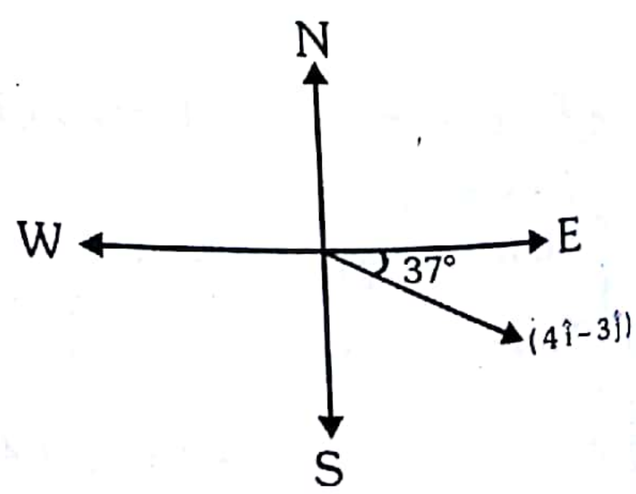
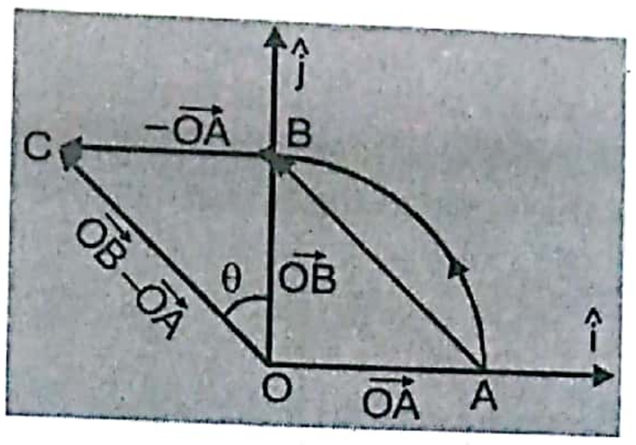


Illustration 3.

A particle goes along a quadrant of a circle of radius 10m from A to B as shown in fig. Find the magnitude of displacement and distance along the path AB, and angle between displacement vector and x-axis ?



Solution

Displacement $\vec{AB} = \vec{OB} - \vec{OA} = (10\hat{j} - 10\hat{i})$m

$|\vec{AB}| = \sqrt{10^2 + 10^2} = 10\sqrt{2}$ m

From ΔOBC $\tan\theta = \dfrac{OA}{OB} = \dfrac{10}{10} = 1 \quad \Rightarrow \quad \theta = 45°$

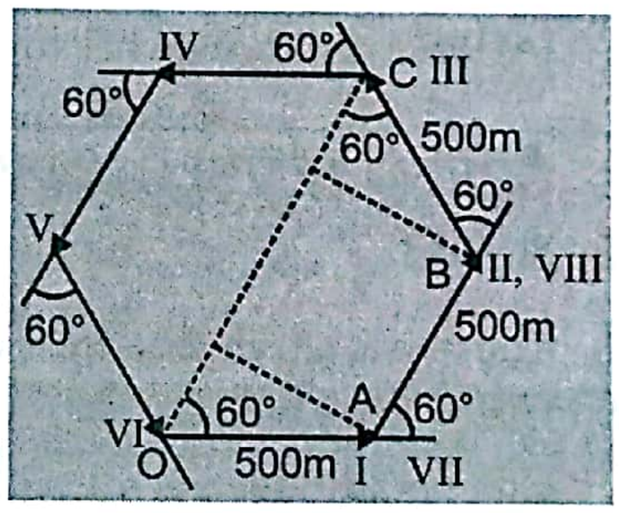
Angle between displacement vector $\vec{OC}$ and x-axis = 90° + 45° = 135°

Distance of path AB = $\dfrac{1}{4}$ (circumference) = $\dfrac{1}{4}(2\pi R)$ m = (5π) m

Illustration 4.

On an open ground a motorist follows a track that turns to his left by an angle of 60° after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of displacement with the total path length covered by the motorist in each case.

Solution



At III turn

$|\text{Displacement}| = |\vec{OA} + \vec{AB} + \vec{BC}| = |\vec{OC}|$

$= 500 \cos 60° + 500 + 500 \cos 60°$

$= 500 \times \dfrac{1}{2} + 500 + 500 \times \dfrac{1}{2} = 1000$ m

So $|\text{Displacement}| = 1000$ m from O to C

Distance $= 500 + 500 + 500 = 1500$ m $\quad \therefore \quad \dfrac{|\text{Displacement}|}{\text{Distance}} = \dfrac{1000}{1500} = \dfrac{2}{3}$

At VI turn

$\because$ initial and final positions are same so $|\text{displacement}| = 0$ and distance = 500 × 6 = 3000 m

$\therefore \quad \dfrac{|\text{Displacement}|}{\text{Distance}} = \dfrac{0}{3000} = 0$

At VIII turn

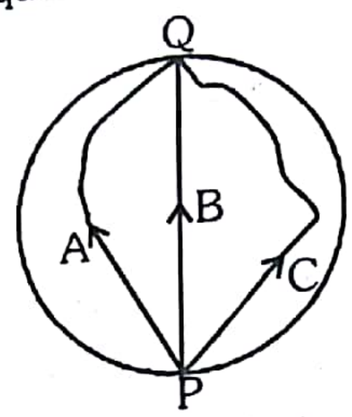
$|\text{Displacement}| = 2(500)\cos\left(\dfrac{60°}{2}\right) = 1000 \times \cos 30° = 1000 \times \dfrac{\sqrt{3}}{2} = 500\sqrt{3}$ m

Distance $= 500 \times 8 = 4000$ m $\quad \therefore \quad \dfrac{|\text{Displacement}|}{\text{Distance}} = \dfrac{500\sqrt{3}}{4000} = \dfrac{\sqrt{3}}{8}$

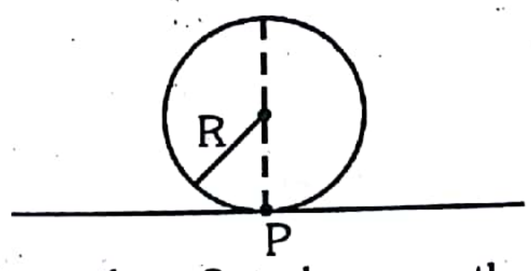
BEGINNER'S BOX-1

1. A particle moves on a circular path of radius 'r', It completes one revolution in 40 s. Calculate distance and displacement in 2 min 20 s.

2. Three girls skating on a circular ice ground of radius 200 m start from a point P on the edge of the ground and reach a point Q diametrically opposite to P following different paths as shown in figure. What is the magnitude of the displacement for each ? For which girl is this equal to the actual length of path skate?



3 A wheel of radius 'R' is placed on ground and its contact point is 'P'. If wheel starts rolling without slipping and completes half a revolution, find the displacement of point P.



4. A man moves 4 m along east direction, then 3m along north direction, after that he climbs up a pole to a height 12m. Find the distance covered by him and his displacement.

5. A person moves on a semicircular track of radius 40 m. If he starts at one end of the track and reaches the other end, find the distance covered and magnitude of displacement of the person.

6. A man has to go 50m due north, 40m due east and 20m due south to reach a cafe from his home. (A) What distance he has to walk to reach the cafe ? (B) What is his displacement from his home to the cafe?

4. SPEED & VELOCITY

4.1 Speed

The rate at which distance is covered with respect to time is called speed. It is a scalar quantity

Dimension : $[M^0L^1T^{-1}]$

Unit : m/s (S.I.), cm/s (C.G.S.)

Note : For a moving particle speed can never be negative or zero, it is always positive.

Uniform speed

When a particle covers equal distances in equal intervals of time, (no matter how small the intervals are) then it is said to be moving with uniform speed.

$$\text{Uniform speed} = \frac{\text{Distance}}{\text{Time}}$$

Non-uniform (variable) speed

In non-uniform speed particle covers unequal distances in equal intervals of time.

Average speed : The average speed of a particle for a given 'interval of time' is defined as the ratio of total distance travelled to the time taken.

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Time taken}} \quad \text{i.e. } v_{av} = \frac{\Delta s}{\Delta t}$$

GOLDEN KEY POINTS

- When a particle moves with different uniform speeds $v_1, v_2, v_3 \ldots\ldots v_4$ in different time intervals $t_1, t_2, t_3, \ldots t_n$ respectively, its average speed over the total time of journey is given as

$$v_{av} = \frac{\text{Total distance covered}}{\text{Total time elapsed}} = \frac{s_1 + s_2 + s_3 + \ldots\ldots + s_n}{t_1 + t_2 + t_3 + \ldots\ldots + t_n} = \frac{v_1t_1 + v_2t_2 + v_3t_3 + \ldots\ldots}{t_1 + t_2 + t_3 + \ldots\ldots}$$

If $t_1 = t_2 = t_3 = \ldots\ldots = t_n$ then

$$v_{av} = \frac{v_1 + v_2 + v_3 + \ldots\ldots + v_n}{n} \quad \text{(Arithmetic mean of speeds)}$$

- When a particle describes different distances $s_1, s_2, s_3, \ldots\ldots s_n$ with speeds $v_1, v_2, v_3 \ldots\ldots v_n$ respectively then the average speed of particle over the total distance will be given as

$$v_{av} = \frac{\text{Total distance covered}}{\text{Total time elapsed}} = \frac{s_1 + s_2 + s_3 + \ldots\ldots + s_n}{t_1 + t_2 + t_3 + \ldots\ldots + t_n} = \frac{s_1 + s_2 + s_3 + \ldots\ldots + s_n}{\frac{s_1}{v_1} + \frac{s_2}{v_2} + \frac{s_3}{v_3} + \ldots\ldots + \frac{s_n}{v_n}}$$

If $s_1 = s_2 = s_3 = \ldots\ldots = s_n$ then

$$v_{av} = \frac{n}{\frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \ldots\ldots + \frac{1}{v_n}} \quad \text{(Harmonic mean of speeds)}$$

Instantaneous speed

It is the speed of a particle at a particular instant of time.

Instantaneous speed $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$

4.2 Velocity

The rate of change of position *i.e.* rate of displacement with time is called velocity.

It is a vector quantity

Dimension : $[M^0L^1T^{-1}]$

Unit : m/s (S.I.), cm/s (C.G.S.)

GOLDEN KEY POINTS

- Velocity may be positive, negative or zero.
- Direction of velocity (instantaneous) is always in the direction of change in position.
- Speedometer measures the instantaneous speed of a vehicle.

Uniform velocity

A particle is said to have uniform velocity, if magnitude as well as direction of its velocity remain same. This is possible only when it moves in a straight line without reversing its direction.

Non-uniform velocity

A particle is said to have non-uniform velocity, if both either magnitude or direction of velocity change.

Average velocity

It is defined as the ratio of displacement to time taken by the body

Average velocity $= \frac{\text{Displacement}}{\text{Time taken}}$; $\quad \vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$

Its direction is along the displacement.

GOLDEN KEY POINTS

- If velocity is continuously changing with time i.e. velocity is the function of time then time average velocity

$$< v >_t = \frac{\int v\, dt}{\int dt}$$

- If velocity is continuously changing with distance i.e. velocity is the function of space (distance) then space average velocity :-

$$< v >_s = \frac{\int v\, ds}{\int ds}$$

- Average speed ≥ | Average velocity |

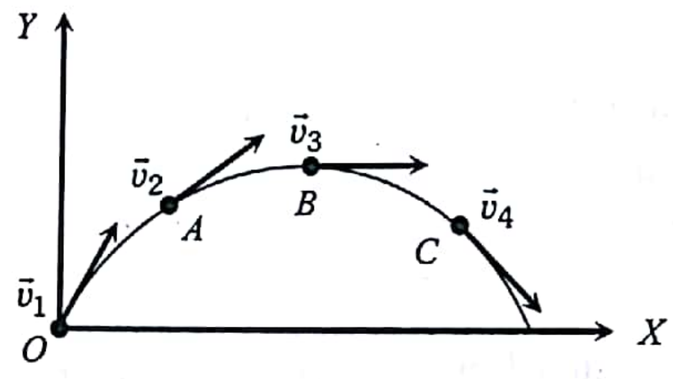
Instantaneous velocity

It is the velocity of a particle at a particular instant of time.

Instantaneous velocity $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$

GOLDEN KEY POINTS

- The direction of instantaneous velocity is always tangential to the path followed by the particle.



- When a particle is moving on any path, the magnitude of instantaneous velocity is equal to the instantaneous speed.
- A particle may have constant speed but variable velocity.

  Example : When a particle is performing uniform circular motion then for every instant of its circular motion its speed remains constant but velocity changes at every instant.
- When particle moves with uniform velocity then its instantaneous speed ,magnitude of instantaneous velocity, average speed and magnitude of average velocity are all equal.

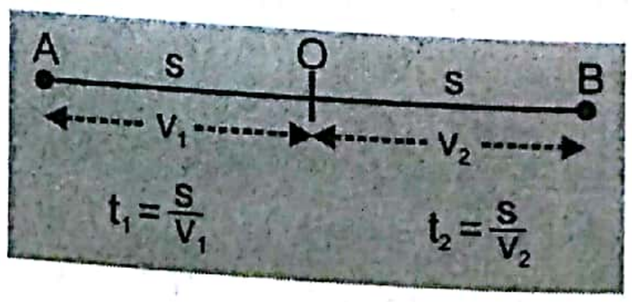
Illustrations

Illustration 5.

If a particle travels the first half distance with speed v_1 and second half distance with speed v_2.Find its average speed during the journey.

Solution

$$v_{av} = \frac{s+s}{t_1+t_2} = \frac{2s}{\frac{s}{v_1}+\frac{s}{v_2}} = \frac{2v_1v_2}{v_1+v_2}$$



Note :- Here v_{av} is the harmonic mean of two speeds.

Illustration 18.

A train, travelling at 20 km/hr is approaching a platform. A bird is sitting on a pole on the platform. When the train is at a distance of 2 km from pole, brakes are applied which produce a uniform deceleration in it. At that instant the bird flies towards the train at 60 km/hr and after touching the nearest point on the train flies back to the pole and then flies towards the train and continues repeating itself. Calculate how much distance the bird covers before the train stops ?

Solution

For retardation of train $v^2 = u^2 + 2as \Rightarrow 0 = (20)^2 + 2(a)(2) \Rightarrow a = -100$ km/hr^2

Time required to stop the train $v = u + at \Rightarrow 0 = 20 - 100t \Rightarrow t = \frac{1}{5}$ hr

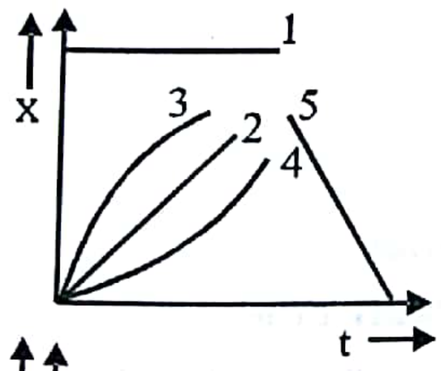
For Bird, speed $= \frac{\text{Distance}}{\text{time}} \Rightarrow s_B = v_B \times t = 60 \times \frac{1}{5} = 12$ km.

BEGINNER'S BOX-4

1. A particle starts from rest, moves with constant acceleration for 15s. If it covers s_1 distance in first 5s then distance s_2 in next 10s, then find the relation between s_1 & s_2.
2. The engine of a train passes an electric pole with a velocity 'u' and the last compartment of the train crosses the same pole with a velocity v. Then find the velocity with which the mid-point of the train passes the pole. Assume acceleration to be uniform.
3. A bullet losses 1/n of its velocity in passing through a plank. What is the least number of planks required to stop the bullet ? (Assuming constant retardation)
4. A car moving along a straight highway with speed 126 km h^{-1} is brought to a halt within a distance of 200m. What is the retardation of the car (assumed uniform) and how long does it take for the car to stop?
5. A car is moving with speed u. Driver of the car sees red traffic light. His reaction time is t, then find out the distance travelled by the car after the instant when the driver decided to apply brakes. Assume uniform retardation 'a' after applying brakes.
6. If a body starts from rest and travels 120cm in the 6th second then what is the acceleration ?

7. GRAPHICAL SECTION

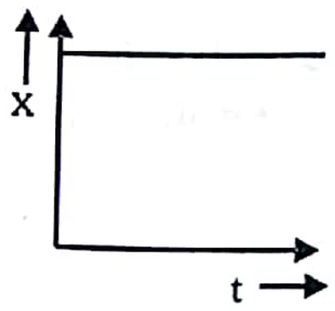
Position - time graph



Slope of this graph represents instantaneous velocity.

$\because \quad \tan\theta = \frac{\text{displacement}}{\text{time}} = \text{velocity}$

(i)



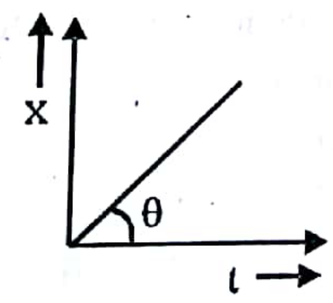
$\theta = 0°$

$\tan\theta = \tan 0° = 0$

velocity = 0

i.e. body is at rest.

(ii)



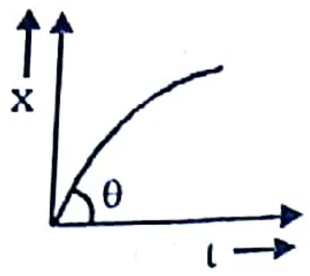
θ = constant

$\tan\theta$ = constant

velocity = constant

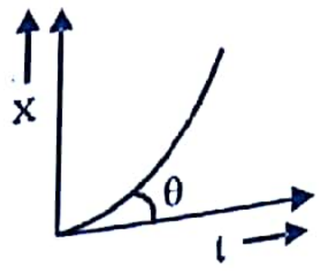
i.e. the body is in uniform motion

(iii)



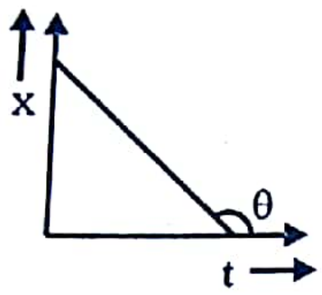
θ is decreasing with time
∴ tanθ is decreasing with time
∴ velocity is decreasing with time
i.e. non uniform motion

(iv)



θ is increasing with time
∴ tanθ is increasing with time
∴ velocity is increasing with time
i.e. non uniform motion

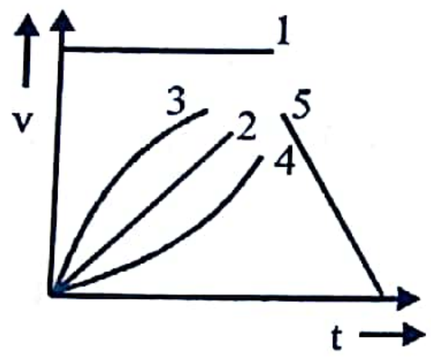
(v)



$\theta > 90°$
$\tan\theta = -ve$
velocity = –ve but constant
i.e. uniform motion

Area of x-t graph = $\int x dt$ = No physical significance

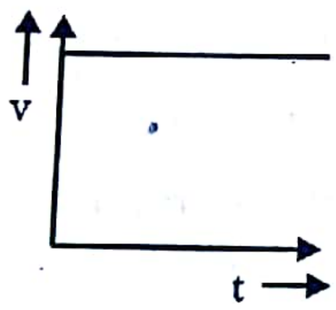
Velocity time graph



Slope of this graph represents acceleration.

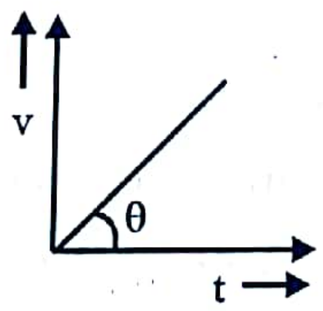
$\because \tan\theta = \frac{\text{velocity}}{\text{time}}$ = acceleration

(i)



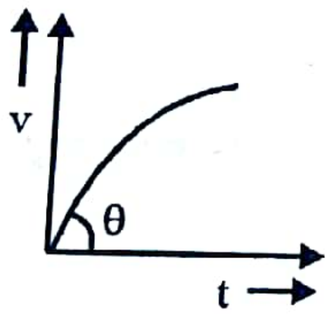
$\theta = 0°$
$\tan\theta = \tan 0° = 0$
acceleration = 0
i.e. v = constant or uniform motion

(ii)



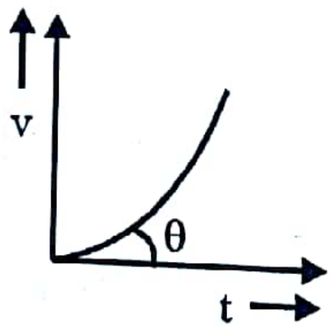
θ = constant
tanθ = constant
acceleration = constant
i.e. uniformly accelerated motion

(iii)



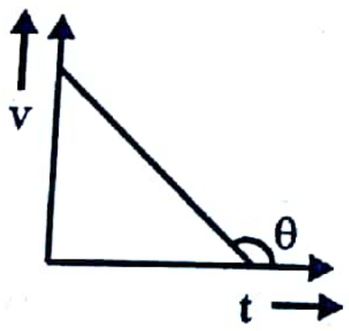
θ is decreasing with time
∴ tanθ is decreasing with time
∴ acceleration is decreasing with time
i.e. acceleration goes on decreasing with time but it is not retardation

(iv)



θ is increasing with time
∴ tanθ is increasing with time
∴ acceleration is increasing with time
i.e. acceleration goes on increasing with time

(v)



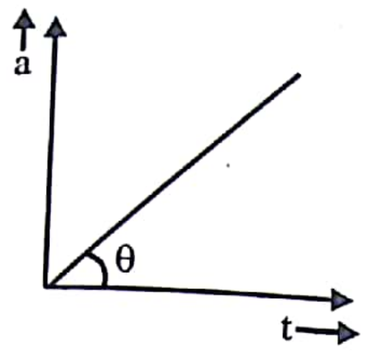
$\theta > 90°$
$\tan\theta = -ve$
acceleration = –ve but constant
i.e. constant or uniform retardation is acting on the body

Area of v-t graph = $\int v\, dt$ = displacement = change in position

Acceleration-time graph

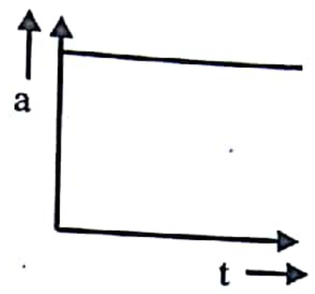
Area of a–t graph = $\int a\,dt = \int dv = v_2 - v_1$ = change in velocity

(i)



i.e. uniformly increasing acceleration.

(ii)

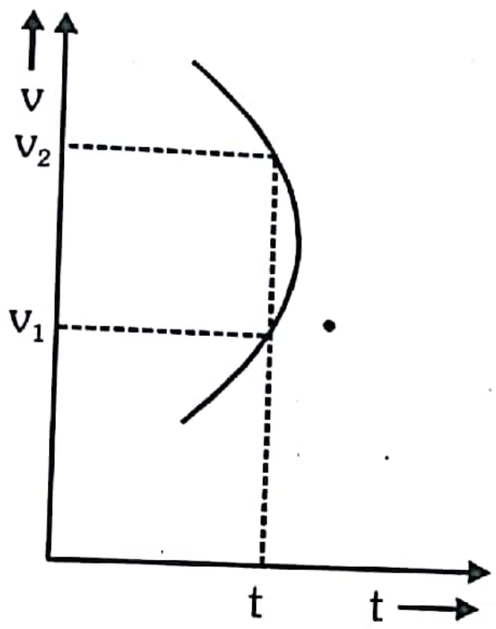


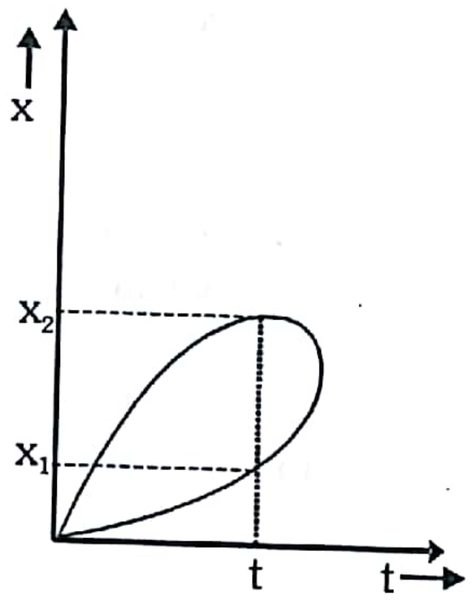
$a \propto t^0$ i.e. uniform or constant acceleration

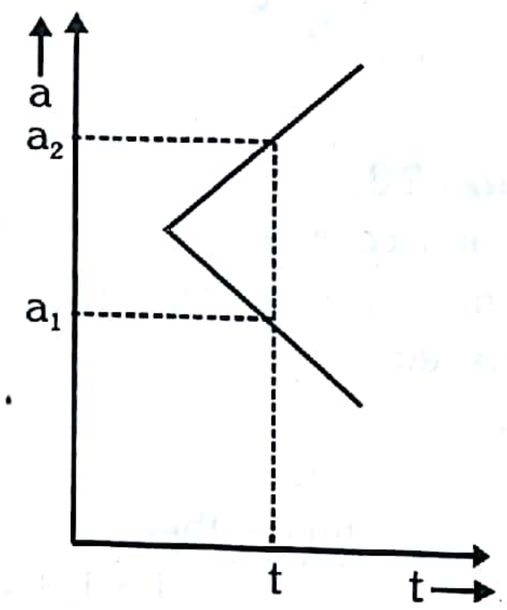
GOLDEN KEY POINTS

- Total area enclosed between speed-time (v-t) graph and time axis represent distance.
- Vector sum of total area enclosed between v-t graph and time axis represent displacement.
- Following graphs do not exist in practice :

Case-I

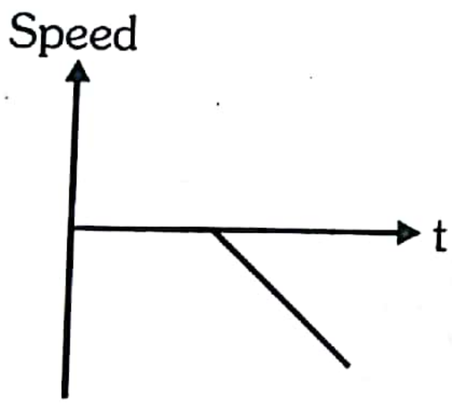


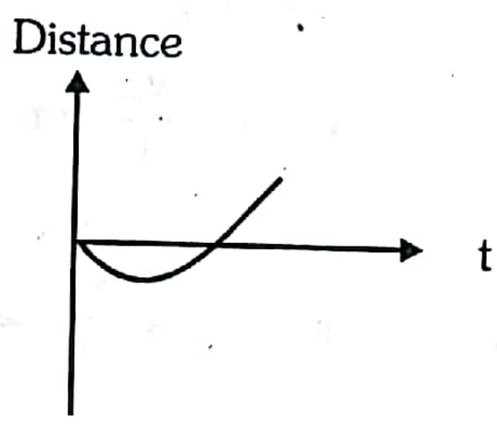




Explanation : In practice, at any instant body can not have two velocities or displacements or accelerations simultaneously.

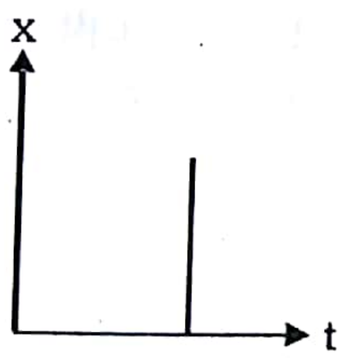
Case- II

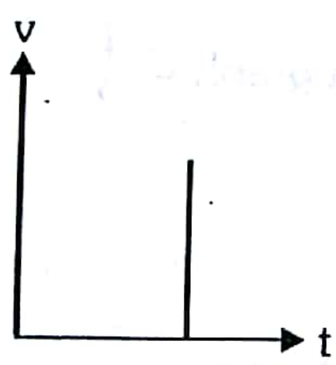


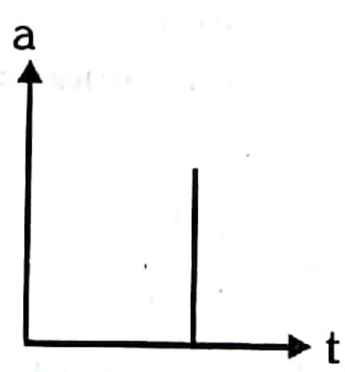


Explanation : Speed or distance can never be negative.

Case - III







Explanation : It is not possible to change any quantity without consuming time i.e. time can't be constant.

BASIC MATHEMATICS USED IN PHYSICS

Mathematics is the supporting tool of Physics. Elementary knowledge of basic mathematics is useful in problem solving in Physics. In this chapter we study *Elementary Algebra, Trigonometry, Coordinate Geometry* and *Calculus (differentiation and integration).*

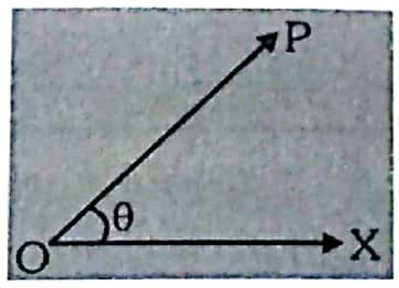
1. TRIGONOMETRY

1.1 Angle

Consider a revolving line OP.

Suppose that it revolves in anticlockwise direction starting from its initial position OX .

The angle is defined as the amount of revolution that the revolving line makes with its initial position.



From fig. the angle covered by the revolving line OP is $\theta = \angle POX$

The angle

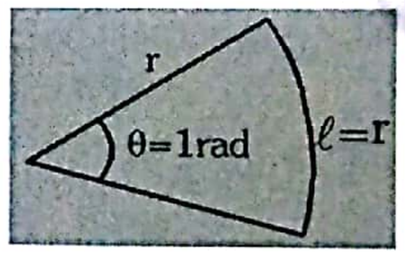
is taken **positive** if it is traced by the revolving line in anticlockwise direction and

is taken **negative** if it is covered in clockwise direction.

$$1^\circ = 60' \text{ (minute)}$$
$$1' = 60'' \text{ (second)}$$

1 right angle = 90° (degrees) also 1 right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle subtended at the centre of a circle by an arc of the circle, whose length is equal to the radius of the circle. 1 rad = $\frac{180^\circ}{\pi} \approx 57.3^\circ$



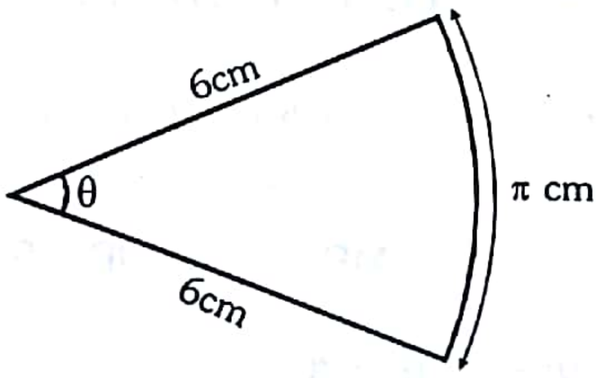
To convert an angle from degree to radian multiply it by $\frac{\pi}{180^\circ}$

To convert an angle from radian to degree multiply it by $\frac{180^\circ}{\pi}$

Illustrations

Illustration 1.

A circular arc is of length π cm. Find angle subtended by it at the centre in radian and degree.



Solution

$$\theta = \frac{s}{r} = \frac{\pi \text{ cm}}{6 \text{ cm}} = \frac{\pi}{6}\text{rad} = 30^\circ \quad \text{As 1 rad} = \frac{180^\circ}{\pi} \quad \text{So } \theta = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$$

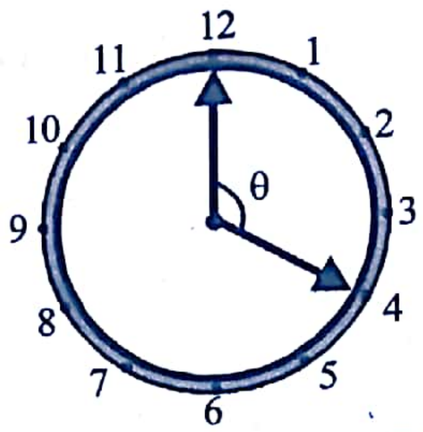
Illustration 2.

When a clock shows 4 o'clock, how much angle do its minute and hour needles make?

(1) 120° (2) $\frac{\pi}{3}$ rad (3) $\frac{2\pi}{3}$ rad (4) 160°

Ans. (1,3)

Solution



From diagram angle $\theta = 4 \times 30° = 120° = \frac{2\pi}{3}$ rad

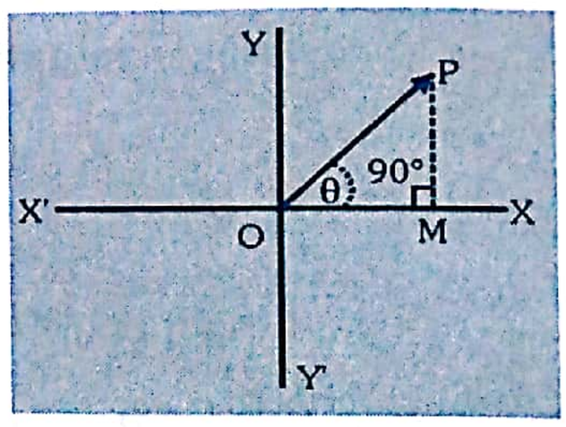
1.2 Trigonometrical ratios (or T ratios)

Let two fixed lines XOX' and YOY' intersect at right angles to each other at point O. Then,

(i) Point O is called origin.

(ii) XOX' is known as X-axis and YOY' as Y-axis.

(iii) Portions XOY, YOX', X'OY' and Y'OX are called I, II, III and IV quadrant respectively.



Consider that the revolving line OP has traced out angle θ (in I quadrant) in anticlockwise direction. From P, draw perpendicular PM on OX. Then, side OP (in front of right angle) is called hypotenuse, side MP (in front of angle θ) is called **opposite side or perpendicular** and side OM (making angle θ with hypotenuse) is called **adjacent side or base.**

The three sides of a right angled triangle are connected to each other through six different ratios, called trigonometric ratios or simply T-ratios :

$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{MP}{OP} \qquad \cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP}$$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{MP}{OM} \qquad \cot\theta = \frac{\text{base}}{\text{perpendicular}} = \frac{OM}{MP}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM} \qquad \text{cosec}\,\theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{MP}$$

It can be easily proved that :

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

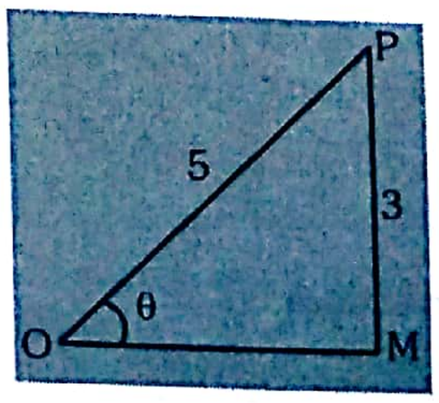
$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \text{cosec}^2\theta$$

Illustrations

Illustration 3.

Given sin θ = 3/5. Find all the other T-ratios, if θ lies in the first quadrant.

Solution



In Δ OMP, $\sin\theta = \frac{3}{5}$ so MP = 3 and OP = 5

$\because$ $OM = \sqrt{(5)^2 - (3)^2} = \sqrt{25-9} = \sqrt{16} = 4$

Now, $\cos\theta = \frac{OM}{OP} = \frac{4}{5}$ $\tan\theta = \frac{MP}{OM} = \frac{3}{4}$

$\cot\theta = \frac{OM}{MP} = \frac{4}{3}$ $\sec\theta = \frac{OP}{OM} = \frac{5}{4}$ $\text{cosec}\,\theta = \frac{OP}{MP} = \frac{5}{3}$

Table : The T-ratios of a few standard angles ranging from 0° to 180°

Angle (θ)	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0

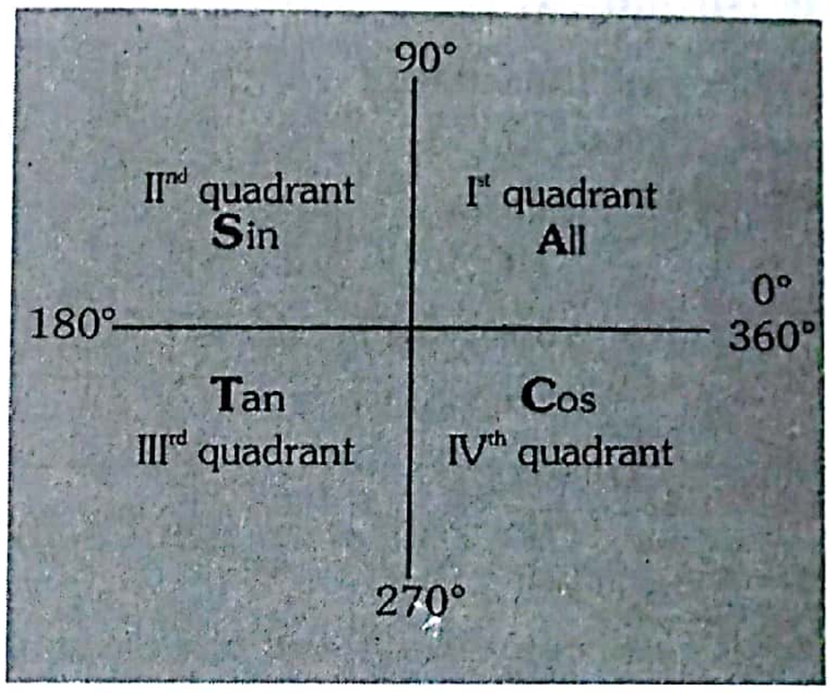
1.3 Four Quadrants and ASTC Rule*

In first quadrant, all trigonometric ratios are positive.

In second quadrant, only sinθ and cosecθ are positive.

In third quadrant, only tanθ and cotθ are positive.

In fourth quadrant, only cosθ and secθ are positive



* Remember as Add Sugar To Coffee or After School To College.

1.4 Trigonometrical Ratios of General Angles (Reduction Formulae)

(i) Trigonometric function of an angle (2nπ + θ) where n=0, 1, 2, 3,.... will be remain same.

$\sin(2n\pi+\theta) = \sin\theta$ $\quad$ $\cos(2n\pi+\theta) = \cos\theta$ $\quad$ $\tan(2n\pi+\theta)=\tan\theta$

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will remain same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

$\sin(\pi-\theta) = +\sin\theta$ $\quad$ $\cos(\pi-\theta) = -\cos\theta$ $\quad$ $\tan(\pi-\theta)= -\tan\theta$

$\sin(\pi+\theta)= -\sin\theta$ $\quad$ $\cos(\pi+\theta)= -\cos\theta$ $\quad$ $\tan(\pi+\theta)= +\tan\theta$

$\sin(2\pi-\theta)= -\sin\theta$ $\quad$ $\cos(2\pi-\theta)= +\cos\theta$ $\quad$ $\tan(2\pi-\theta)= -\tan\theta$

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$\sin\left(\frac{\pi}{2}+\theta\right) = +\cos\theta$ $\quad$ $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$ $\quad$ $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$ $\quad$ $\cos\left(\frac{\pi}{2}-\theta\right) = +\sin\theta$ $\quad$ $\tan\left(\frac{\pi}{2}-\theta\right) = +\cot\theta$

(iv) Trigonometric function of an angle –θ (negative angles)

$\sin(-\theta) = -\sin\theta$ $\quad$ $\cos(-\theta) = +\cos\theta$ $\quad$ $\tan(-\theta) = -\tan\theta$

$\sin(90° + \theta) = \cos\theta$ $\cos(90° + \theta) = -\sin\theta$ $\tan(90° + \theta) = -\cot\theta$	$\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta) = -\cos\theta$ $\tan(180° - \theta) = -\tan\theta$	$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$	$\sin(90° - \theta) = \cos\theta$ $\cos(90° - \theta) = \sin\theta$ $\tan(90° - \theta) = \cot\theta$
$\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$ $\tan(180° + \theta) = \tan\theta$	$\sin(270° - \theta) = -\cos\theta$ $\cos(270° - \theta) = -\sin\theta$ $\tan(270° - \theta) = \cot\theta$	$\sin(270° + \theta) = -\cos\theta$ $\cos(270° + \theta) = \sin\theta$ $\tan(270° + \theta) = -\cot\theta$	$\sin(360° - \theta) = -\sin\theta$ $\cos(360° - \theta) = \cos\theta$ $\tan(360° - \theta) = -\tan\theta$

Illustrations

Illustration 4.

Find the value of
(i) $\cos(-60°)$ (ii) $\tan 210°$ (iii) $\sin 300°$ (iv) $\cos 120°$

Solution

(i) $\cos(-60°) = \cos 60° = \frac{1}{2}$

(ii) $\tan 210° = \tan(180° + 30°) = \tan 30° = \frac{1}{\sqrt{3}}$

(iii) $\sin 300° = \sin(270° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

(iv) $\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$

BEGINNER'S BOX-1

1. Find the values of :

(i) $\tan(-30°)$ (ii) $\sin 120°$ (iii) $\sin 135°$ (iv) $\cos 150°$ (v) $\sin 270°$ (vi) $\cos 270°$

2. If $\sec\theta = \frac{5}{3}$ and $0 < \theta < \frac{\pi}{2}$. Find all the other T-ratios.

1.5 A few important trigonometric formulae

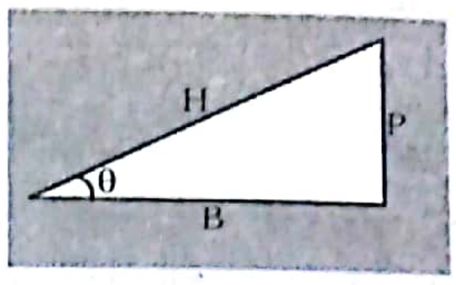
$\sin(A + B) = \sin A\cos B + \cos A\sin B$	$\cos(A + B) = \cos A\cos B - \sin A\sin B$
$\sin(A - B) = \sin A\cos B - \cos A\sin B$	$\cos(A - B) = \cos A\cos B + \sin A\sin B$
$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$	$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$
$\sin 2A = 2\sin A\cos A$	$\cos 2A = \cos^2 A - \sin^2 A$
$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$	$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
	$1 + \cos A = 2\cos^2\frac{A}{2}$, $1 - \cos A = 2\sin^2\frac{A}{2}$

1.6 Range of trigonometric functions

As $\sin\theta = \frac{P}{H}$ and $P \le H$ so $-1 \le \sin\theta \le 1$

As $\cos\theta = \frac{B}{H}$ and $B \le H$ so $-1 \le \cos\theta \le 1$

As $\tan\theta = \frac{P}{B}$ so $-\infty < \tan\theta < \infty$



Remember : $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

1.7 Small Angle Approximation

If θ is small (say $< 5°$) then $\sin\theta \approx \theta$, $\cos\theta \approx 1$ & $\tan\theta \approx \theta$. Here θ must be in radians.

Illustrations

Illustration 5.

Find the approximate values of (i) $\sin 1°$ (ii) $\tan 2°$ (iii) $\cos 1°$

Solution

(i) $\sin 1° = \sin\left(1° \times \frac{\pi}{180°}\right) = \sin\left(\frac{\pi}{180}\right) \approx \frac{\pi}{180}$ (ii) $\tan 2° = \tan\left(2° \times \frac{\pi}{180°}\right) = \tan\left(\frac{\pi}{90}\right) \approx \frac{\pi}{90}$ (iii) $\cos 1° \approx 1$

2. COORDINATE GEOMETRY

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If a point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

- **Origin**

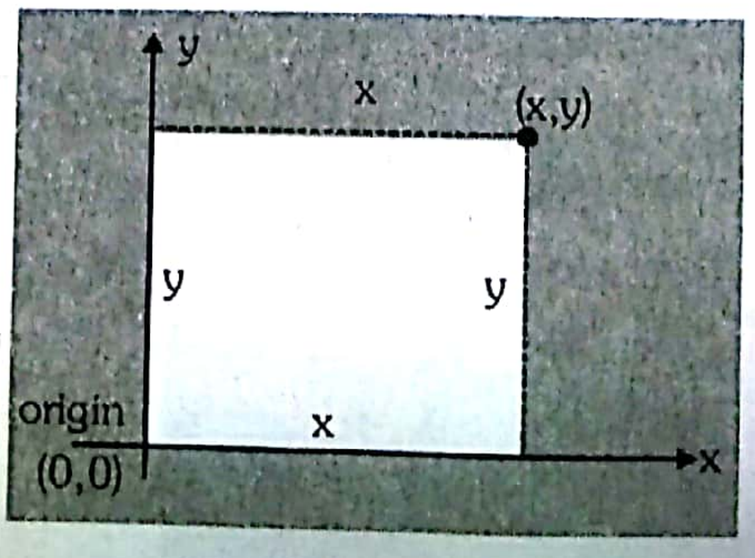
This is any fixed point which is convenient to you. All measurements are taken w.r.t. this fixed point.

- **Axis or Axes**

Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.

2.1 Position of a point in xy plane

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure. Here x-coordinate and y-coordinate is called abscissa and ordinate respectively.



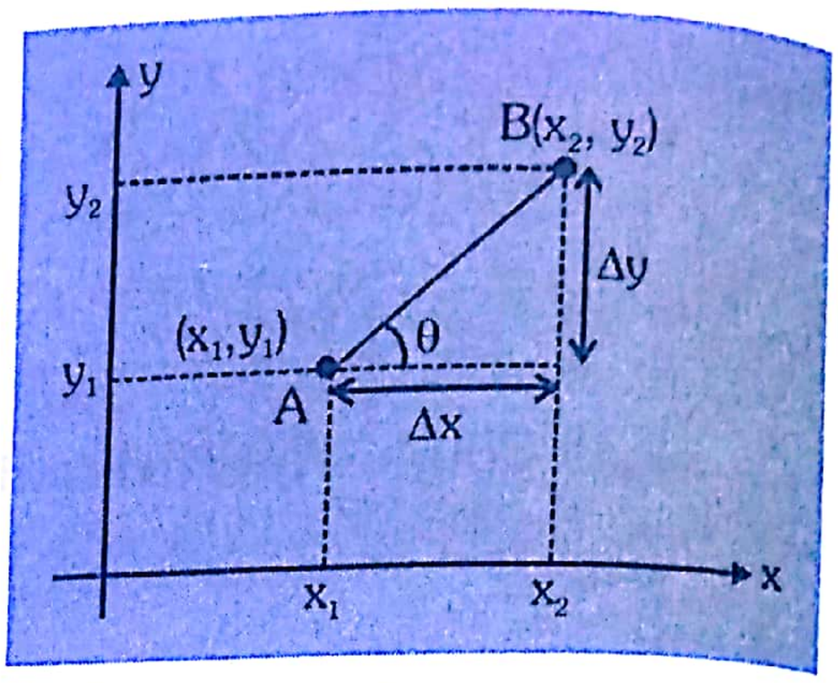
2.2 Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note : In space $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

2.3 Slope of a Line

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$ [If both axes have identical scales]

Here θ is the angle made by line with positive x-axis.

Slope of a line is a quantitative measure of inclination.

Illustrations

Illustration 6.

For point (2, 14) find abscissa and ordinate. Also find distance from y and x-axis.

Solution

Abscissa = x-coordinate = 2 = distance from y-axis.

Ordinate = y-coordinate = 14 = distance from x-axis.

Illustration 7.

Find value of a if distance between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

Solution

By using distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \Rightarrow 13 = \sqrt{[3-(-9)]^2 + [3-a]^2}$

$\Rightarrow 13^2 = 12^2 + (3-a)^2 \Rightarrow (3-a)^2 = 13^2 - 12^2 = 5^2 \Rightarrow (3-a) = \pm 5 \Rightarrow a = -2$ cm or 8 cm

Illustration 8.

A dog wants to catch a cat. The dog follows the path whose equation is $y-x = 0$ while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are :

(1) (2, –2) (2) (2, 2) (3) (–2, 2) (4) (–2, –2)

Ans. (2,4)

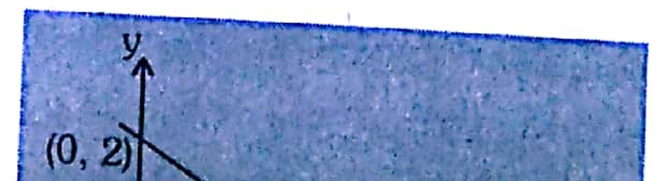
Solution

Let catching point be (x_1, y_1) then, $y_1 - x_1 = 0$ and $x_1^2 + y_1^2 = 8$

Therefore, $2x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$; So possible points are (2, 2) and (–2, –2).

BEGINNER'S BOX-2

1. Distance between two points (8, – 4) and (0, a) is 10. All the values are in the same unit of length. Find the positive value of a.
2. Calculate the distance between two points (0, –1, 1) and (3, 3, 13).
3. Calculate slope of shown line



3. DIFFERENTIATION

3.1 Function

Constant: A quantity, whose value remains unchanged during mathematical operations, is called a constant quantity. The integers, fractions like π ,e etc are all constants.

Variable: A quantity, which can take different values, is called a variable quantity. A variable is usually represented as x, y, z, etc.

Function: A quantity y is called a function of a variable x, if corresponding to any given value of x, there exists a single definite value of y. The phrase 'y is function of x' is represented as $y = f(x)$

For example, consider that y is a function of the variable x which is given by $y = 3x^2 + 7x + 2$

If $x = 1$, then $y = 3(1)^2 + 7(1) + 2 = 12$ and when $x = 2$, $y = 3(2)^2 + 7(2) + 2 = 28$

Therefore, when the value of variable x is changed, the value of the function y also changes but corresponding to each value of x, we get a single definite value of y. Hence, $y = 3x^2 + 7x + 2$ represents a function of x.

3.2 Physical meaning of $\frac{dy}{dx}$

(I) The ratio of small change in the function y and the variable x is called the average rate of change of y w.r.t. x.

For example, if a body covers a small distance Δs in small time Δt, then

average velocity of the body, $v_{av} = \frac{\Delta s}{\Delta t}$

Also, if the velocity of a body changes by a small amount Δv in small time Δt, then average acceleration of the body, $a_{av} = \frac{\Delta v}{\Delta t}$

(II) When $\Delta x \to 0$ The limiting value of $\frac{\Delta y}{\Delta x}$ is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

It is called the instantaneous rate of change of y w.r.t. x.

The *differentiation of a function w.r.t. a variable implies the instantaneous rate of change of the function w.r.t. that variable.*

Like wise, instantaneous velocity of the body $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$

and instantaneous acceleration of the body $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$

3.3 Theorems of differentiation

1. If c = constant, $\frac{d}{dx}(c) = 0$

2. $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = \frac{d}{dx}(cu) = c\frac{du}{dx}$

3. $y = u \pm v \pm w$, where, u, v and w are functions of x, $\frac{dy}{dx} = \frac{d}{dx}(u \pm v \pm w) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$

4. $y = uv$ where u and v are functions of x, $\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

5. $y = \frac{u}{v}$, where u and v are functions of x, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

6. $y = x^n$, n real number, $\frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$

Illustrations

Illustration 9.

Find $\frac{dy}{dx}$, when (i) $y = \sqrt{x}$ (ii) $y = x^5 + x^4 + 7$ (iii) $y = x^2 + 4x^{-1/2} - 3x^{-2}$

Solution

(i) $y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

(ii) $y = x^5 + x^4 + 7 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^5 + x^4 + 7) = \frac{d}{dx}(x^5) + \frac{d}{dx}(x^4) + \frac{d}{dx}(7)$

$= 5x^4 + 4x^3 + 0 = 5x^4 + 4x^3$

(iii) $y = x^2 + 4x^{-1/2} - 3x^{-2} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 + 4x^{-1/2} - 3x^{-2}) = \frac{d}{dx}(x^2) + \frac{d}{dx}(4x^{-1/2}) - \frac{d}{dx}(3x^{-2})$

$= \frac{d}{dx}(x^2) + 4\frac{d}{dx}(x^{-1/2}) - 3\frac{d}{dx}(x^{-2}) = 2x + 4\left(-\frac{1}{2}\right)x^{-3/2} - 3(-2)x^{-3}$

$= 2x - 2x^{-3/2} + 6x^{-3}$

BEGINNER'S BOX-3

1. Find $\frac{dy}{dx}$ for the following :

   (i) $y = x^{7/2}$ (ii) $y = x^{-3}$ (iii) $y = x$ (iv) $y = x^5 + x^3 + 4x^{1/2} + 7$

   (v) $y = 5x^4 + 6x^{3/2} + 9x$ (vi) $y = ax^2 + bx + c$ (vii) $y = 3x^5 - 3x - \frac{1}{x}$

2. Given $s = t^2 + 5t + 3$, find $\frac{ds}{dt}$.

3. If $s = ut + \frac{1}{2}at^2$, where u and a are constants. Obtain the value of $\frac{ds}{dt}$.

4. The area of a blot of ink is growing such that after t seconds, its area is given by $A = (3t^2+7)\ cm^2$. Calculate the rate of increase of area at t=5 second.

5. The area of a circle is given by $A = \pi r^2$, where r is the radius. Calculate the rate of increase of area w.r.t. radius.

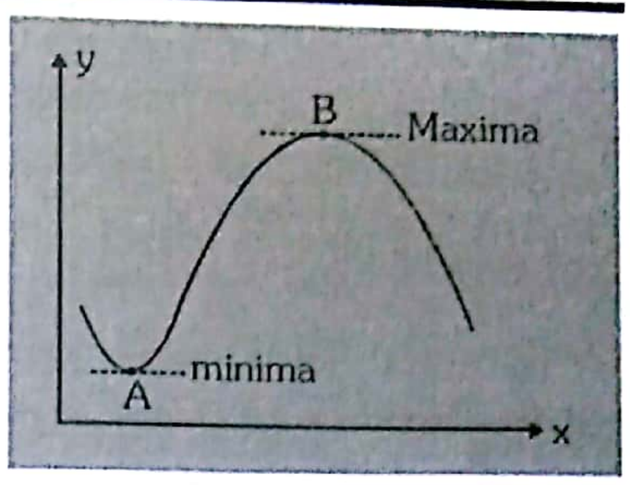
6. Obtain the differential coefficient of the following :

   (i) $(x - 1)(2x + 5)$ (ii) $\frac{1}{2x+1}$ (iii) $\frac{3x+4}{4x+5}$ (iv) $\frac{x^2}{x^3+1}$

3.4 Formulae for differential coefficients of trigonometric, logarithmic and exponential functions

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$

3.5 Maximum and Minimum value of a Function



Higher order derivatives are used to find the maximum and minimum values of a function. At the points of maxima and minima, first derivative (i.e. $\frac{dy}{dx}$) becomes zero.

At point 'A' (minima) :

As we see in figure, in the neighbourhood of A, slope increases so $\frac{d^2y}{dx^2} > 0$.

Condition for minima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0}$

At point 'B' (maxima) : As we see in figure, in the neighbourhood of B, slope decreases so $\frac{d^2y}{dx^2} < 0$

Condition for maxima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0}$

Illustrations

Illustration 10.

The minimum value of $y = 5x^2 - 2x + 1$ is

(1) $\frac{1}{5}$ (2) $\frac{2}{5}$ (3) $\frac{4}{5}$ (4) $\frac{3}{5}$

Ans. (3)

Solution

For maximum/minimum value $\frac{dy}{dx} = 0 \Rightarrow 5(2x) - 2(1) + 0 = 0 \Rightarrow x = \frac{1}{5}$. Now at $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = 10$ which is positive

so y has minimum value at $x = \frac{1}{5}$. Therefore $y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 = \frac{4}{5}$

4. INTEGRATION

In integral calculus, the differential coefficient of a function is given. We are required to find the function. Integration is basically used for summation . Σ is used for summation of discrete values, while $\int$ sign is used for continous function.

If I is integration of f(x) with respect to x then $I = \int f(x)\,dx$ [we can check $\frac{dI}{dx} = f(x)$] $\because \int f'(x)\,dx = f(x) + c$

where c = an arbitrary constant

Let us proceed to obtain intergral of x^n w.r.t. x. $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$

Since the process of integration is the reverse process of differentiation,

$$\int (n+1)x^n dx = x^{n+1} \quad \text{or} \quad (n+1)\int x^n dx = x^{n+1} \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1}$$

The above formula holds for all values of n, except $n = -1$.

It is because, for $n = -1$, $\int x^n dx = \int x^{-1} dx = \int \frac{1}{x} dx$

$$\because \quad \frac{d}{dx}(\log_e x) = \frac{1}{x} \qquad \therefore \quad \int \frac{1}{x} dx = \log_e x$$

Similarly, the formulae for integration of some other functions can be obtained if we know the differential coefficients of various functions.

4.1 Few basic formulae of integration

Following are a few basic formulae of integration :

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, Provided $n \neq -1$

2. $\int \sin x \, dx = -\cos x + c$ $\quad (\because \frac{d}{dx}(\cos x) = -\sin x)$

3. $\int \cos x \, dx = \sin x + c$ $\quad (\because \frac{d}{dx}(\sin x) = \cos x)$

4. $\int \frac{1}{x} dx = \log_e x + c$ $\quad (\because \frac{d}{dx}(\log_e x) = \frac{1}{x})$

5. $\int e^x \, dx = e^x + c$ $\quad (\because \frac{d}{dx}(e^x) = e^x)$

Illustrations

Illustration 11.

Integrate w.r.t. x. : (i) $x^{11/2}$ (ii) x^{-7} (iii) $x^{p/q} (p/q \neq -1)$

Solution

(i) $\int x^{11/2} dx = \frac{x^{11/2+1}}{\frac{11}{2}+1} + c = \frac{2}{13} x^{13/2} + c$ (ii) $\int x^{-7} dx = \frac{x^{-7+1}}{-7+1} + c = -\frac{1}{6} x^{-6} + c$ (iii) $\int x^{\frac{p}{q}} dx = \frac{x^{\frac{p}{q}+1}}{\frac{p}{q}+1} + c = \frac{q}{p+q} x^{(p+q)/q} + c$

Illustration 12.

Evaluate $\int \left(x^2 - \cos x + \frac{1}{x} \right) dx$

Solution

$$I = \int x^2 dx - \int \cos x dx + \int \frac{1}{x} dx = \frac{x^{2+1}}{2+1} - \sin x + \log_e x + c = \frac{x^3}{3} - \sin x + \log_e x + c$$

BEGINNER'S BOX-4

1. Evaluate the following integrals :

(i) $\int x^{15} dx$ (ii) $\int x^{-3/2} dx$ (iii) $\int (3x^{-7} + x^{-1}) dx$ (iv) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx$

(v) $\int \left(x + \frac{1}{x} \right) dx$ (vi) $\int \left(\frac{a}{x^2} + \frac{b}{x} \right) dx$ (a and b are constant)

4.2 Definite Integrals

When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

If $\frac{d}{dx}(f(x)) = f'(x)$, then

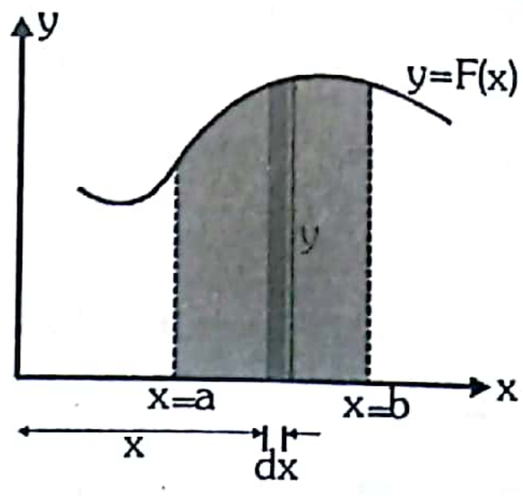
$\int f'(x) dx$ is called indefinite integral and $\int_a^b f'(x) dx$ is called definite integral

Here, a and b are called lower and upper limits of the variable x.

After carrying out integration, the result is evaluated between upper and lower limits as explained below :

$$\int_a^b f'(x) dx = |f(x)|_a^b = f(b) - f(a)$$

4.3 Area under a curve and definite integration



Area of small shown darkly shaded element = ydx = f(x) dx

If we sum up all areas between x=a and x= b then

$$\int_a^b f(x)dx = \text{shaded area between curve and x-axis.}$$

Illustrations

Illustration 13.

The integral $\int_1^5 x^2dx$ is equal to

(1) $\frac{125}{3}$ (2) $\frac{124}{3}$ (3) $\frac{1}{3}$ (4) 45

Solution **Ans. (2)**

$$\int_1^5 x^2dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3} - \frac{1^3}{3}\right] = \frac{125}{3} - \frac{1}{3} = \frac{124}{3}$$

BEGINNER'S BOX-5

1. Evaluate the following integrals

(i) $\int_R^\infty \frac{GMm}{x^2}dx$ **(ii)** $\int_{r_1}^{r_2} -k\frac{q_1q_2}{x^2}dx$ **(iii)** $\int_u^v Mvdv$ **(iv)** $\int_0^\infty x^{-1/2}dx$

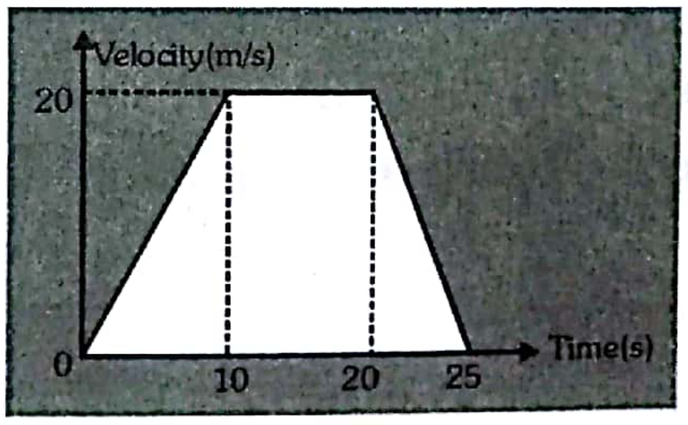
(v) $\int_0^{\pi/2} \sin x\,dx$ **(vi)** $\int_0^{\pi/2} \cos x\,dx$ **(vii)** $\int_{-\pi/2}^{\pi/2} \cos x\,dx$

4.4 Average value of a continuous function in an interval

Average value of a function y = f(x), over an interval $a \le x \le b$ is given by $y_{av} = \frac{\int_a^b ydx}{\int_a^b dx} = \frac{\int_a^b ydx}{b-a}$

Illustrations

Illustration 14.



The velocity-time graph of a car moving along a straight road is shown in figure. The average velocity of the car in first 25 seconds is

(1) 20 m/s (2) 14 m/s

(3) 10 m/s (4) 17.5 m/s

Solution : **Ans. (2)**

$$\text{Average velocity} = \frac{\int_0^{25} vdt}{25-0} = \frac{\text{Area of v-t graph between t=0 to t = 25 s}}{25} = \frac{1}{25}\left[\left(\frac{25+10}{2}\right)(20)\right] = 14 \text{ m/s}$$

Illustration 15.

Determine the average value of $y = 2x + 3$ in the interval $0 \le x \le 1$.

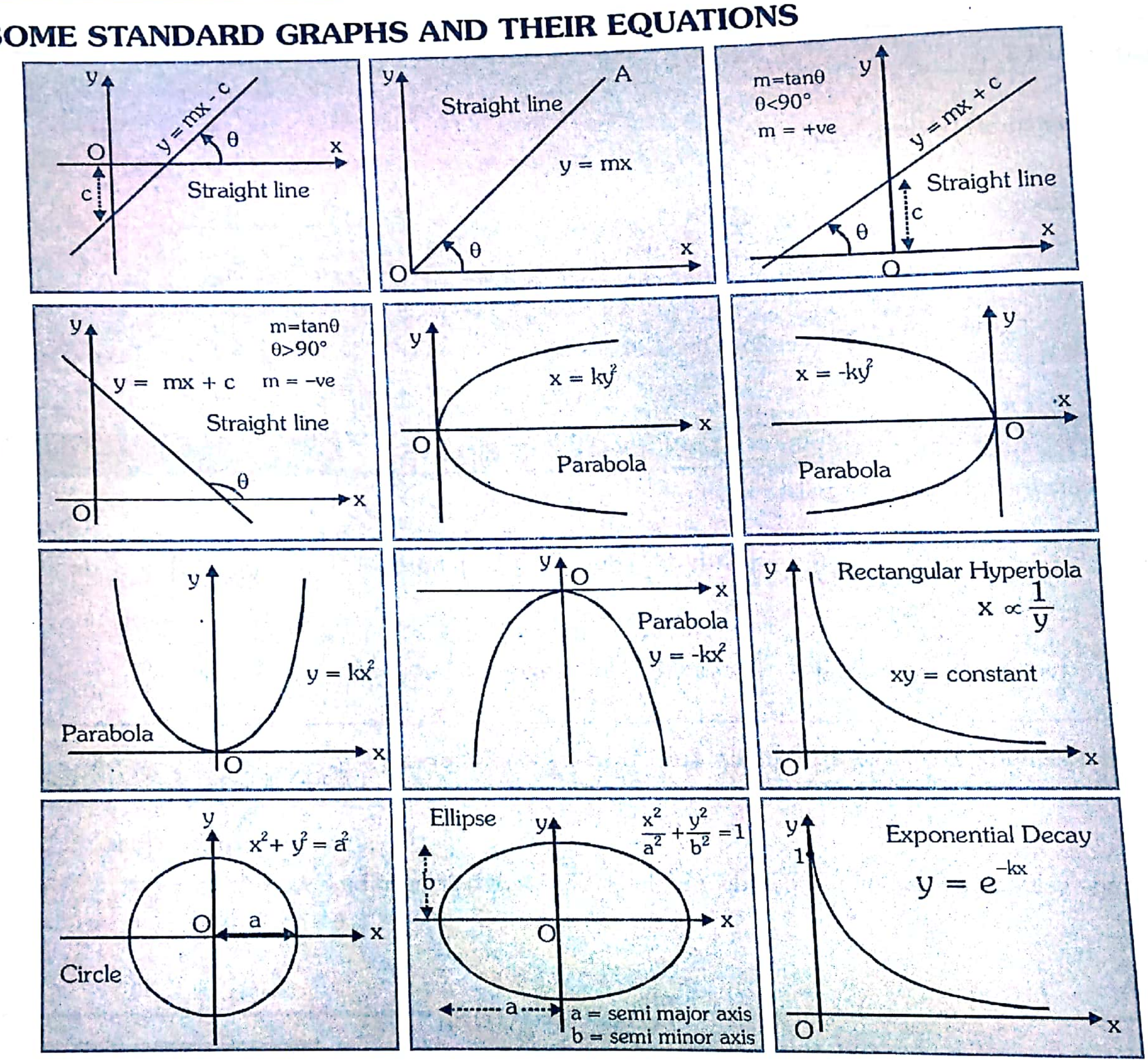
(1) 1 (2) 5 (3) 3 (4) 4

Ans. (4)

Solution

$$y_{av} = \frac{\int_0^1 y\,dx}{1-0} = \int_0^1 (2x+3)\,dx = \left[2\left(\frac{x^2}{2}\right) + 3x\right]_0^1 = 1^2 + 3(1) - 0^2 - 3(0) = 1 + 3 = 4$$

5. SOME STANDARD GRAPHS AND THEIR EQUATIONS



6. ALGEBRA

6.1 Quadratic equation and its solution :

An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation.

Equation $ax^2 + bx + c = 0$ is the general quadratic equation.

The general solution of the above quadratic equation or value of variable x is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Sum of roots $= x_1 + x_2 = -\frac{b}{a}$ and product of roots $= x_1 x_2 = \frac{c}{a}$

For real roots $b^2 - 4ac \ge 0$ and for imaginary roots $b^2 - 4ac < 0$

Illustrations

Illustration 16.

Solve the equation $2x^2 + 5x - 12 = 0$

Solution

By comparison with the standard quadratic equation $a = 2$, $b = 5$ and $c = -12$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4 \times 2 \times (-12)}}{2 \times 2} = \frac{-5 \pm \sqrt{121}}{4} = \frac{-5 \pm 11}{4} = \frac{+6}{4}, \frac{-16}{4} \quad \text{or} \quad x = \frac{3}{2}, -4$$

Illustration 17.

The speed (v) of a particle moving along a straight line is given by $v = t^2 + 3t - 4$ where v is in m/s and t in seconds. Find time t at which the particle will momentarily come to rest.

Solution

When particle comes to rest, $v = 0$.

So $t^2 + 3t - 4 = 0 \Rightarrow t = \frac{-3 \pm \sqrt{9 - 4(1)(-4)}}{2(1)} \Rightarrow t = 1$ or -4

Neglect negative value of t, Hence $t = 1$ s

Illustration 18.

The speed (v) and time (t) for an object moving along straight line are related as $t^2 + 100 = 2vt$ where v is in meter/second and t is in second. Find the possible positive values of v.

Solution

For possible values of v, time t must be real so from $b^2 - 4ac \geq 0$

we have $(-2v)^2 - 4(1)(100) \geq 0$

$\Rightarrow 4v^2 - 400 \geq 0 \Rightarrow v^2 - 100 \geq 0$

$\Rightarrow (v - 10)(v + 10) \geq 0 \Rightarrow v \geq 10$ and $v \leq -10$

Hence possible positive values of v are $v \geq 10$ m/s.

BEGINNER'S BOX-6

1. Solve for x : (i) $10x^2 - 27x + 5 = 0$ (ii) $pqx^2 - (p^2 + q^2)x + pq = 0$

2. In quadratic equation $ax^2 + bx + c = 0$, if discriminant is $D = b^2 - 4ac$, then roots of the quadratic equation are : (choose the correct alternative)

(1) Real and distinct, if $D > 0$
(2) Real and equal (i.e., repeated roots), if $D = 0$.
(3) Non-real (i.e. imaginary), if $D < 0$
(4) All of the above are correct

6.2 Binomial Expression :

An algebraic expression containing two terms is called a binomial expression.

For example $(a+b)$, $(a+b)^3$, $(2x-3y)^{-1}$, $\left(x + \frac{1}{y}\right)$ etc. are binomial expressions.

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2 \times 1}a^{n-2}b^2 + \ldots\ldots, \qquad (1+x)^n = 1 + nx + \frac{n(n-1)}{2 \times 1}x^2 + \ldots\ldots$$

Binomial Approximation

If x is very small, compared to 1, then terms containing higher powers of x can be neglected so $(1+x)^n \approx 1 + nx$

Illustrations

Illustration 19.

Calculate $\sqrt{0.99}$

Solution

$$\sqrt{0.99} = (1-0.01)^{1/2} \approx 1 - \frac{1}{2}(0.01) \approx 1 - 0.005 \approx 0.995$$

Illustration 20.

The mass m of a body moving with a velocity v is given by $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ where m_0 = rest mass of body = 10 kg and c = speed of light = 3×10^8 m/s. Find the value of m at v = 3×10^7 m/s.

Solution

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 10\left[1-\left(\frac{3\times10^7}{3\times10^8}\right)^2\right]^{-1/2} = 10\left[1-\frac{1}{100}\right]^{-1/2} \approx 10\left[1-\left(-\frac{1}{2}\right)\left(\frac{1}{100}\right)\right] = 10+\frac{10}{200} \approx 10.05 \text{ kg}$$

6.3 Logarithm

Common formulae :

- $\log mn = \log m + \log n$
- $\log \dfrac{m}{n} = \log m - \log n$
- $\log m^n = n \log m$
- $\log_e m = 2.303 \log_{10} m$

6.4 Componendo and Dividendo Rule :

If $\dfrac{p}{q} = \dfrac{a}{b}$ then $\dfrac{p+q}{p-q} = \dfrac{a+b}{a-b}$

6.5 Arithmetic progression (AP)

General form : a, a + d, a +2d, ..., a + (n–1)d. Here a = first term, d = common difference

Sum of n terms $S_n = \dfrac{n}{2}[a+a+(n-1)d] = \dfrac{n}{2}[2a+(n-1)d] = \dfrac{n}{2}[I^{st} \text{ term} + n^{th} \text{ term}]$

Illustrations

Illustration 21.

Find the sum of given Arithmetic Progression 4 + 8 + 12 +.....+ 64

(1) 464 (2) 540 (3) 544 (4) 646

Solution

Ans. (3)

Here a = 4, d = 4, n = 16 So, sum = $\dfrac{n}{2}$[First term + last term] = $\dfrac{16}{2}$[4 + 64] = 8(68) = 544

Note :

(i) Sum of first n natural numbers. $S_n = 1 + 2 + 3 + ... + n = \dfrac{n}{2}[1+n] = \dfrac{n(n+1)}{2}$

(ii) Sum of first n squared natural numbers $S_{n^2} = 1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

BEGINNER'S BOX-7

1. Find sum of first 50 natural numbers.

2. Find $1^2 + 2^2 + \ldots\ldots\ldots\ldots + 10^2$.

6.6 Geometric Progression (GP)

General form : a, ar, $ar^2, \ldots, ar^{n-1}$

Here a = first term, r = common ratio

Sum of n terms $S_n = \dfrac{a(1-r^n)}{1-r}$

For $0 \le r < 1$ Sum of ∞ term $S_\infty = \dfrac{a}{1-r}$ ($\because$ $r < 1$ $\therefore$ $r^\infty \to 0$)

Illustrations

Illustration 22.

Find the sum of given series $1 + 2 + 4 + 8 + \ldots\ldots\ldots\ldots + 256$

(1) 510 (2) 511 (3) 512 (4) 513

Solution :

Ans.[2]

Here a = 1, r = 2, n = 9 ($\because$ $256 = 2^8$). So $S_9 = \dfrac{(1)(1-2^9)}{(1-2)} = 2^9 - 1 = 512 - 1 = 511$

Illustration 23.

Find $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots$ upto ∞.

(1) ∞ (2) 1 (3) 2 (4) 1.925

Solution :

Ans.[3]

Here, a = 1, $r = \dfrac{1}{2}$ So, $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{2}} = 2$

BEGINNER'S BOX-8

1. Find $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \dfrac{1}{16} - \dfrac{1}{32} + \ldots\ldots \infty$.

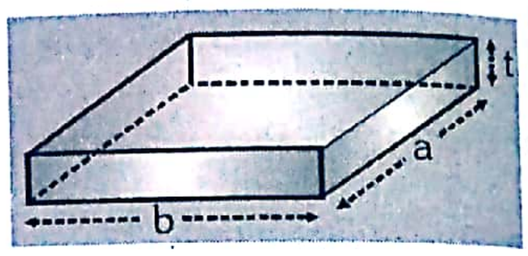
2. Find $F_{net} = GMm\left[\dfrac{1}{r^2} + \dfrac{1}{2r^2} + \dfrac{1}{4r^2} + \ldots\ldots\ldots\ldots \text{up to } \infty\right]$.

7. GEOMETRY

7.1 Formulae for determination of area :

1. Area of a square = $(\text{side})^2$
2. Area of rectangle = length × breadth
3. Area of a triangle = $\frac{1}{2}$ (base × height)
4. Area of trapezoid = $\frac{1}{2}$ (distance between parallel sides) ×(sum of parallel sides)
5. Area enclosed by a circle = πr^2 (r = radius)
6. Surface area of a sphere = $4\pi r^2$ (r = radius)
7. Area of a parallelogram = base × height
8. Area of curved surface of cylinder = $2\pi r\ell$ (r = radius and ℓ = length)
9. Area of ellipse = π ab (a and b are semi major and semi minor axes respectively)
10. Surface area of a cube = $6(\text{side})^2$
11. Total surface area of cone = $\pi r^2 + \pi r\ell$ where $\pi r\ell = \pi r\sqrt{r^2 + h^2}$ = lateral area

7.2 Formulae for determination of volume :



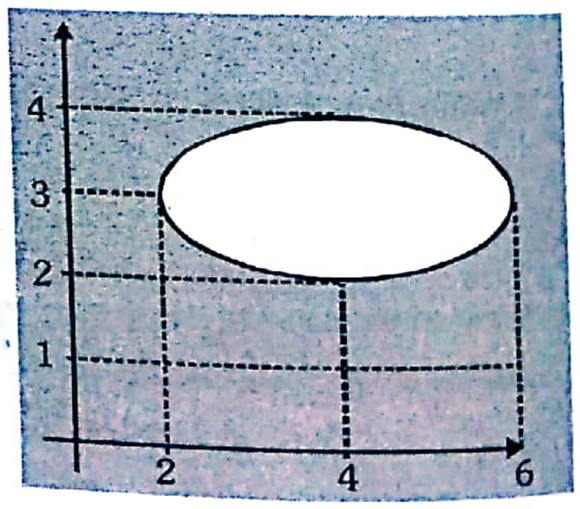
1. Volume of a rectangular slab = length × breadth × height = abt
2. Volume of a cube = $(\text{side})^3$
3. Volume of a sphere = $\frac{4}{3}\pi r^3$ (r = radius)
4. Volume of a cylinder = $\pi r^2\ell$ (r = radius and ℓ is length)
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$ (r = radius and h is height)

Note : $\pi = \frac{22}{7} = 3.14$; $\pi^2 = 9.8776 \approx 10$ and $\frac{1}{\pi} = 0.3182 \approx 0.3$.

Illustrations

Illustration 24.

Calculate the area enclosed by shown ellipse



Solution

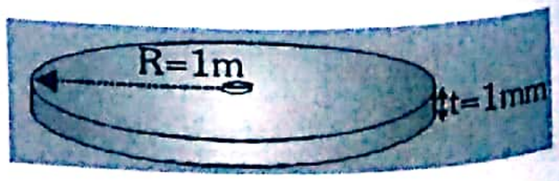
Shaded area = Area of ellipse = πab

Here $a = 6 - 4 = 2$ and $b = 4 - 3 = 1$

$\Rightarrow$ Area = $\pi \times 2 \times 1 = 2\pi$ units

Illustration 25.

Calculate the volume of given disk.



Solution

Volume = $\pi R^2 t = (3.14)\,(1)^2\,(10^{-3}) = 3.14 \times 10^{-3}\ m^3$

VECTORS

Scalar Quantities

A physical quantity which can be described completely by its magnitude only and does not require a direction is known as a scalar quantity.
It obeys the ordinary rules of algebra.

Ex : Distance, mass, time, speed, density, volume, temperature, electric current etc.

Vector Quantities

A physical quantity which requires magnitude and a particular direction, when it is expressed.

Ex. : Displacement, velocity, acceleration, force etc.

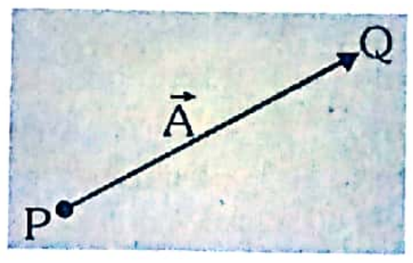
Vector quantities must obey the rules of vector algebra.

A vector is represented by a line headed with an arrow.
Its length is proportional to its magnitude.

$\vec{A}$ is a vector.

$\vec{A} = \overrightarrow{PQ}$

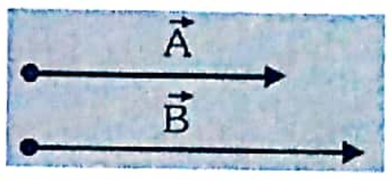
Magnitude of $\vec{A} = |\vec{A}|$ or A



1.1 Types of vector

- **Parallel Vectors :-**

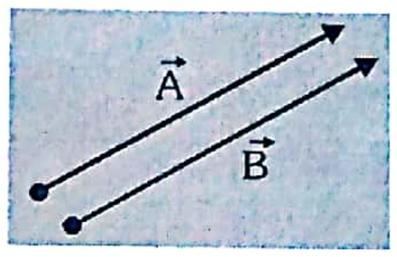
  Those vectors which have same direction are called parallel vectors.
  Angle between two parallel vectors is always 0°



- **Equal Vectors**

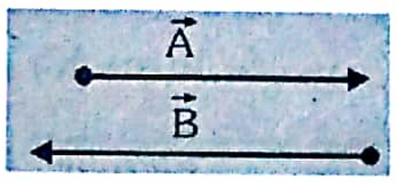
  Vectors which have equal magnitude and same direction are called equal vectors.

  $\vec{A} = \vec{B}$



- **Anti–parallel Vectors :**

  Those vectors which have opposite direction are called anti–parallel vector.
  Angle between two anti–parallel vectors is always 180°

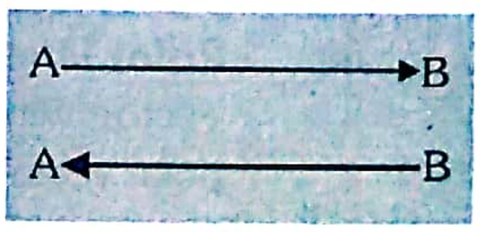


- **Negative (or Opposite) Vectors**

  Vectors which have equal magnitude but opposite direction are called negative vectors of each other.

  $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are negative vectors

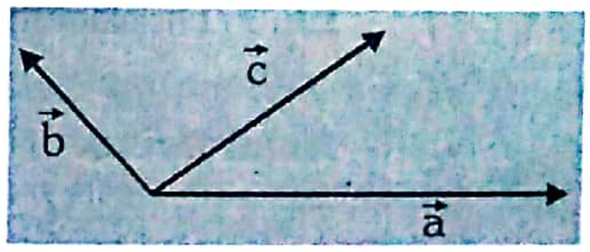
  $\overrightarrow{AB} = -\overrightarrow{BA}$



- **Co-initial vector**

  Co-initial vectors are those vectors which have the same initial point.

  In figure $\vec{a}, \vec{b}$ and $\vec{c}$ are co-initial vectors.



- **Collinear Vectors :**

  The vectors lying in the same line are known as collinear vectors.
  Angle between collinear vectors is either 0° or 180°.

Example.

(i) $\leftarrow \quad \leftarrow$ $(\theta = 0°)$

(ii) $\rightarrow \quad \rightarrow$ $(\theta = 0°)$

(iii) $\leftarrow \quad \rightarrow$ $(\theta = 180°)$

(iv) $\rightarrow \quad \leftarrow$ $(\theta = 180°)$

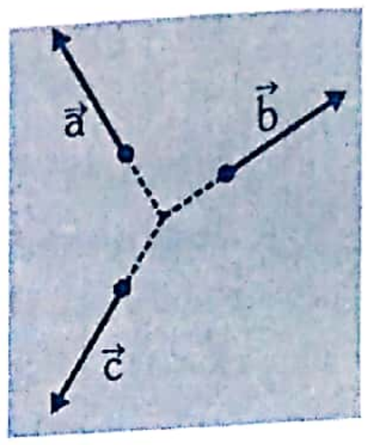
- **Coplanar Vectors**

  Vectors located in the same plane are called coplanar vectors.

  Note :- Two vectors are always coplanar.

- **Concurrent vectors**

  Those vectors which pass through a common point are called concurrent vectors



In figure $\vec{a}, \vec{b}$ and $\vec{c}$ are concurrent vectors.

- **Null or Zero Vector**

  A vector having zero magnitude is called null vector.

  Note : Sum of two vectors is always a vector so, $(\vec{A}) + (-\vec{A}) = \vec{0}$

  $\vec{0}$ is a zero vector or null vector.

- **Unit Vector**

  A vector having unit magnitude is called unit vector. It is used to specify direction. A unit vector is represented by $\hat{A}$ (Read as A cap or A hat or A caret).

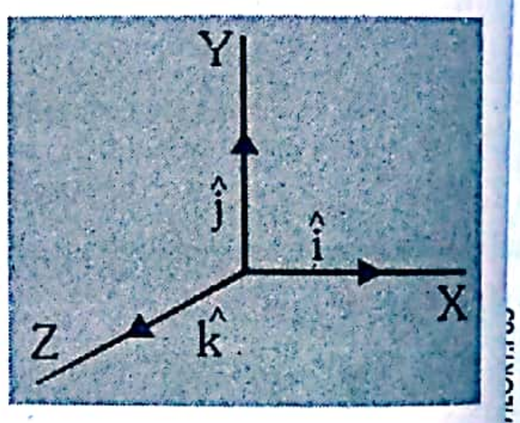
Unit vector in the direction of $\vec{A}$ is $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$ $\left(\text{unit vector} = \dfrac{\text{Vector}}{\text{Magnitude of the vector}}\right)$

$\vec{A} = A\hat{A} = |\vec{A}|\hat{A}$

A unit vector is used to specify the direction of a vector.

Base Vectors

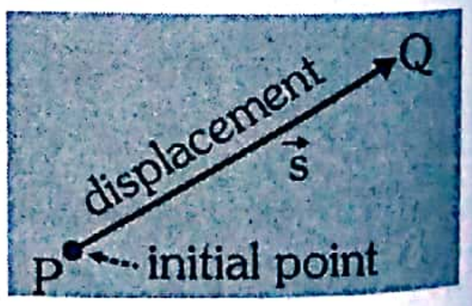
In an XYZ co-ordinate frame there are three unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, these are used to indicate X, Y and Z directions respectively. These three unit vectors are mutually perpendicular to each other.



- **Polar Vector**

  Vectors which have initial point or a point of application are called polar vectors.

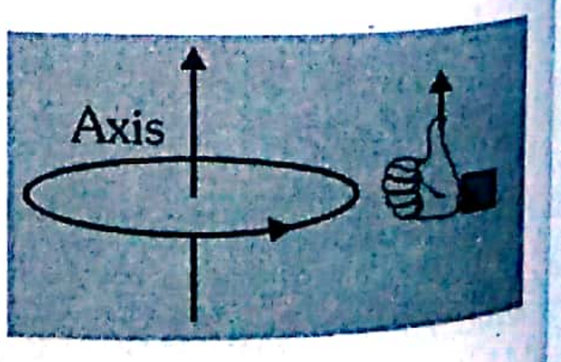
  Ex. : Displacement, force etc.



- **Axial Vector**

  These vectors are used in rotational motion to define rotational effects. Direction of these vectors is always along the axis of rotation in accordance with right hand screw rule or right hand thumb rule.

  Ex. : Infinitesimal angular displacement $(\overrightarrow{d\theta})$, Angular velocity $(\vec{\omega})$, Angular momentum $(\vec{J})$, Angular acceleration $(\vec{\alpha})$ and Torque $(\vec{\tau})$



1.2 Addition of two vectors

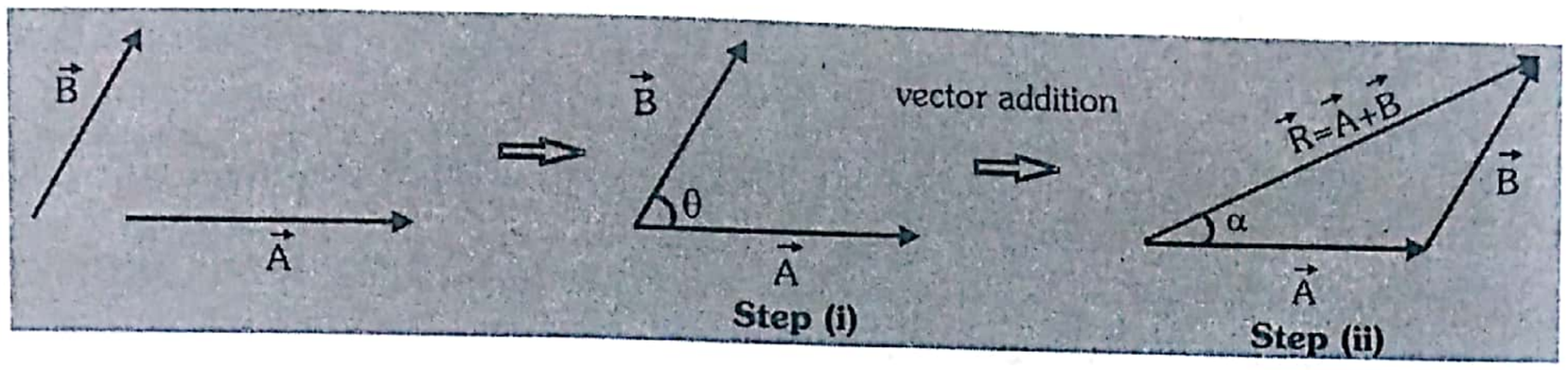
Vector addition can be performed by using following methods

(i) Graphical methods (ii) Analytical methods

Addition of two vectors is quite different from simple algebraic sum of two numbers.

- **Triangle Law of Addition of Two Vectors**

If two vectors are represented by two sides of a triangle in same order then their sum or *'resultant vector'* is given by the third side of the triangle taken in opposite order of the first two vectors.



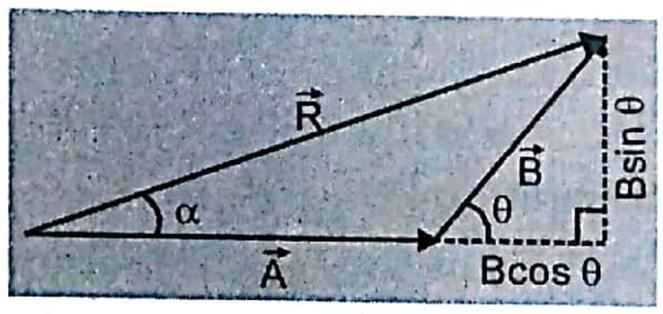
Shift one vector $\vec{B}$, without changing its direction, such that its tail coincide with head of the other vector $\vec{A}$. Now complete the triangle by drawing third side, directed from tail of $\vec{A}$ to head of $\vec{B}$ (it is in opposite order of $\vec{A}$ and $\vec{B}$ vectors).

Sum of two vectors is also called resultant vector of these two vectors. Resultant $\vec{R} = \vec{A} + \vec{B}$

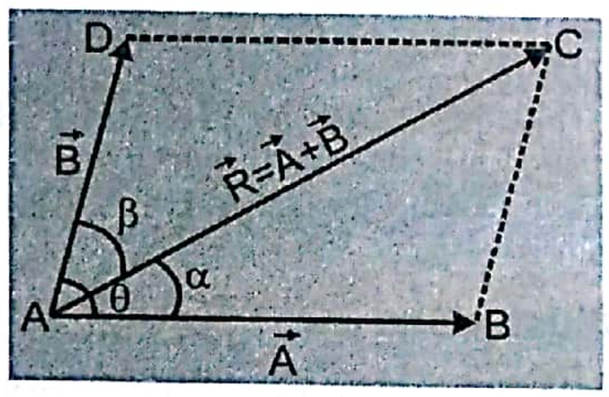
Length of $\vec{R}$ is the magnitude of vector sum i.e. $|\vec{A} + \vec{B}|$

$\therefore\ |\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{(A + B\cos\theta)^2 + (B\sin\theta)^2} = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

Let direction of $\vec{R}$ make angle α with $\vec{A}$: $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$



- **Parallelogram Law of Addition of Two Vectors :** If two vectors are represented by two adjacent sides of a parallelogram which are directed away from their common point then their sum (i.e. resultant vector) is given by the diagonal of the parallelogram passing away through that common point.



$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC} \Rightarrow \vec{A} + \vec{B} = \vec{R}$

$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$, $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$ and $\tan\beta = \dfrac{A\sin\theta}{B + A\cos\theta}$

Illustrations

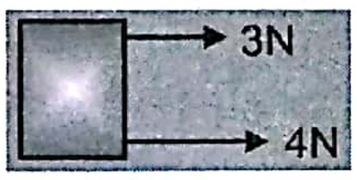
Illustration 26.

Two forces of magnitudes 3N and 4N respectively are acting on a body. Calculate the resultant force if the angle between them is :

(i) 0° (ii) 180° (iii) 90°

Solution

(i)

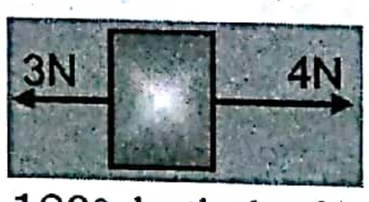


$\theta = 0°$, both the forces are parallel, R = A + B

$\therefore$ Net force or resultant force R = 3 + 4 = 7N

Direction of resultant is along both the forces

(ii)

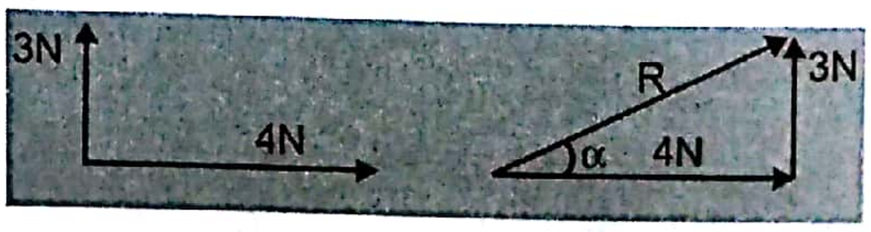


$\theta = 180°$, both the forces are antiparallel, R = A~B

$\therefore$ Net force or resultant force R = 4 – 3 = 1N

Direction of net force is along larger force i.e. along 4N.

(iii) $\theta = 90°$, both the forces are perpendicular

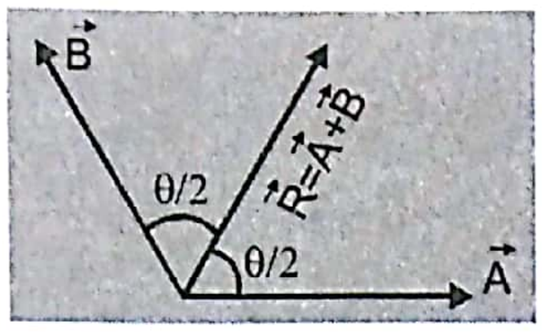


then $R = \sqrt{A^2 + B^2 + 2AB\cos 90°} = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5N$; $\tan\alpha = \frac{3}{4} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 37°$

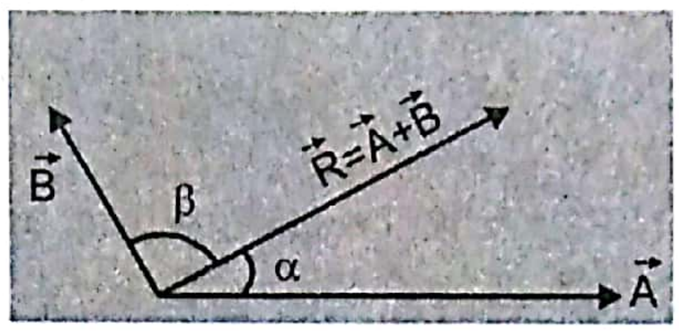
Magnitude of resultant is 5N which is acting at an angle of 37° from 4N force.

- Resultant of two vectors of equal magnitude will be at their bisector.

If $|\vec{A}| = |\vec{B}|$



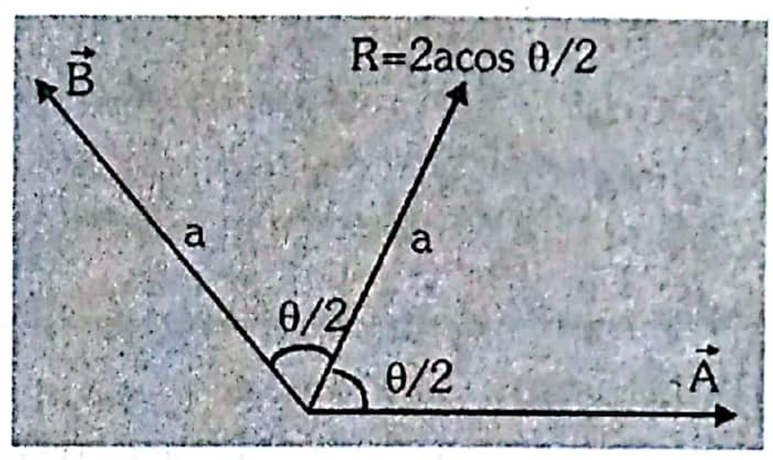
But if $|\vec{A}| > |\vec{B}|$ then angle $\beta > \alpha$



$\vec{R}$ will incline more towards the vector of bigger magnitude.

- If two vectors have equal magnitude i.e. $|\vec{A}| = |\vec{B}| = a$ and angle between them is θ then resultant will be along the bisector of $\vec{A}$ and $\vec{B}$ and its magnitude is equal to $2a\cos\frac{\theta}{2}$

$$|\vec{R}| = |\vec{A}+\vec{B}| = 2a\cos\frac{\theta}{2}$$



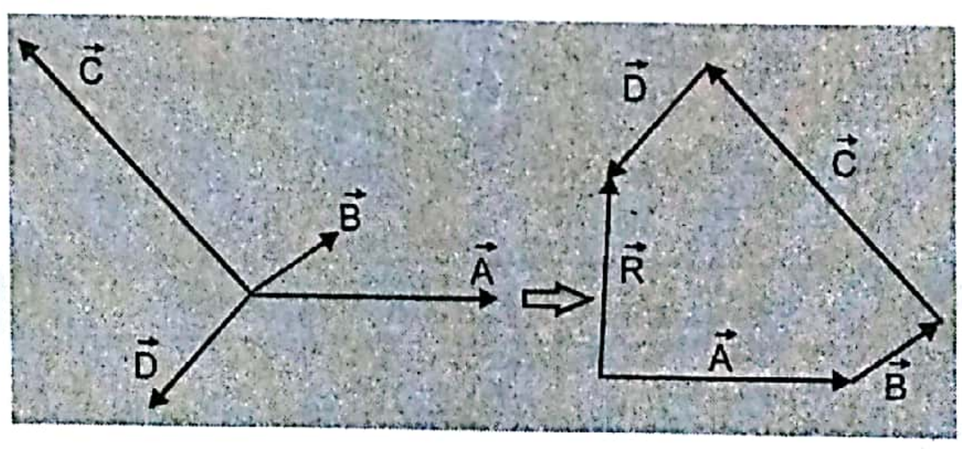
Special Case : If $\theta = 120°$ then $R = 2a\cos\frac{120°}{2} = a$

i.e. If $\theta = 120°$ then $|\vec{R}| = |\vec{A}+\vec{B}| = |\vec{A}| = |\vec{B}| = a$

If resultant of two unit vectors is another unit vector then the angle between them (θ) = 120°.
OR
If the angle between two unit vectors (θ) = 120°, then their resultant is another unit vector.

1.3 Addition of More Than Two Vectors (Law of Polygon)

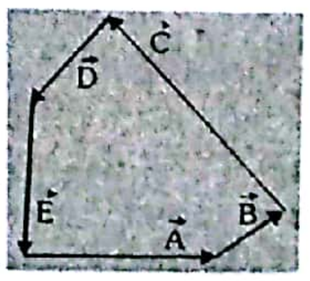
If some vectors are represented by sides of a polygon in same order, then their resultant vector is represented by the closing side of polygon in the opposite order. $\vec{R} = \vec{A}+\vec{B}+\vec{C}+\vec{D}$



GOLDEN KEY POINTS

- In a polygon if all the vectors taken in same order are such that the head of the last vector coincides with the tail of the first vector then their resultant is a null vector.

$$\vec{A}+\vec{B}+\vec{C}+\vec{D}+\vec{E} = \vec{0}$$



- If n coplanar vectors of equal magnitude are arranged at equal angles of separation then their resultant is always zero.

1.4 Subtraction of two vectors

Let $\vec{A}$ and $\vec{B}$ are two vectors. Their difference i.e. $\vec{A}-\vec{B}$ can be treated as sum of the vector $\vec{A}$ and vector $(-\vec{B})$.

$\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$

To subtract $\vec{B}$ from $\vec{A}$, reverse the direction of $\vec{B}$ and add to vector $\vec{A}$ according to law of triangle.

$$|\vec{A}-\vec{B}| = \sqrt{A^2+B^2+2AB\cos(\pi-\theta)} = \sqrt{A^2+B^2-2AB\cos\theta} \ \& \ \tan\alpha = \frac{B\sin\theta}{A-B\cos\theta}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.