Preface

It gives me great pleasure in presenting the new edition of this book. In this edition, the modifications have been dictated by the changes in the CBSE syllabus. The structure and the methods used in the previous editions, which have been appreciated by teachers using the book in class room conditions, remain unchanged.

The main consideration in writing the book was to present the considerable requirements of the syllabus in as simple a manner as possible. Special attention has been paid to the gradation of problems. This will help students gain confidence in problem-solving.

One problem faced by students is the lack of a comprehensive and carefully selected set of solved problems in textbooks of this kind. I have given due weightage to this aspect. Each set of solved-examples is followed by a comprehensive exercise section in which students will get enough questions for practice. Hints have been given to the more difficult questions. Students should take their help as a last resort.

I have received many suggestions and letters of appreciation from teachers all over the country. I thank them all for contributing to the improvement of the book and for their encouragement. I hope they will like this edition as well. And as always, I would like to hear their views on the book.

R S Aggarwal

Mathematics Syllabus

For Class 12

UNIT I. Relations and Functions

1. **Relations and Functions:**

   Types of relations: reflexive, symmetric, transitive and equivalence relations. One to one and onto functions, composite functions, inverse of a function. Binary operations.

2. **Inverse Trigonometric Functions:**

   Definition, range, domain, principal value branches. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

UNIT II. Algebra

1. **Matrices:**

   Concept, notation, order, equality, types of matrices, zero matrix, transpose of a matrix, symmetric and skew symmetric matrices. Addition, multiplication and scalar multiplication of matrices, simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. **Determinants:**

   Determinant of a square matrix (up to 3×3 matrices), properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

UNIT III. Calculus

1. **Continuity and Differentiability:**

   Continuity and differentiability, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivative of implicit function. Concept of exponential and logarithmic functions and their derivative. Logarithmic differentiation. Derivative of functions expressed in parametric forms. Second order derivatives. Rolle's and

Lagrange's Mean Value Theorems (without proof) and their geometric interpretations.

2. **Applications of Derivatives:**
   Applications of derivatives: rate of change, increasing/decreasing functions, tangents and normals, approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

3. **Integrals:**
   Integration as inverse process of differentiation. Integration of a variaty of functions by substitution, by partial fractions and by parts, only simple integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{\sqrt{ax^2 + bx^2 + c}}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{(px+q)}{ax^2 + bx + c} dx, \int \frac{(px+q)}{\sqrt{ax^2 + bx + c}} dx, \int \sqrt{a^2 \pm x^2}\, dx \text{ and } \sqrt{x^2 - a^2}\, dx$$

   to be evaluated.
   Definite integrals as a limit of a sum, Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

4. **Applications of the Integrals:**
   Applications in finding the area under simple curves, especially lines, areas of circles/parabolas/ellipses (in standard form only), area between the two above said curves (the region should be clearly identifiable).

5. **Differential Equations:**
   Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables, homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

$$\frac{dy}{dx} + p(x)y = q(x), \text{ where } p(x) \text{ and } q(x) \text{ are functions of } x.$$

UNIT IV. Vectors and Three-dimensional Geometry

1. **Vectors:**
   Vectors and scalars, magnitude and direction of a vector. Direction cosines/ratios of vectors. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment

ration. Scalar (dot) product of vectors, projection of a vector on a line. Vector (cross product of vectors.

2. **Three-dimensional Geometry:**

   Direction cosines/ratios of a line joining two points. Cartesian and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (i) two lines, (ii) two planes, (iii) a line and a plane. Distance of a point from a plane.

UNIT V. Linear Programming

1. **Linear Programming:**

   Introduction, definition of related terminology such as constraints, objective function, optimization, different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions, feasible and infeasible solutions, optional feasible solutions (up to three non-trivial constrains).

UNIT VI. Probability

1. **Probability:**

   Multiplication theorem on probability. Conditional probability, independent events, total probability, Baye's theorem, Random variable and its probability distribution, mean and variance of haphazard variable Repeated independent (Bernoulli) trials and Binomial distribution.

Weightage

Topic	Marks
1. Relations and Functions	10
2. Algebra	13
3. Calculus	44
4. Vectors & 3-Dimensional Geometry	17
5. Linear Programming	06
6. Probability	10
Total	100

1. RELATIONS

In class XI we discussed about the Cartesian product of sets. Now, we extend our ideas to relation in a set and then in next chapter we shall be taking up functions.

RELATION IN A SET

A relation R in a set A is a subset of $A \times A$.

Thus, *R is a relation in a set $A \Leftrightarrow R \subseteq A \times A$.*

If $(a, b) \in R$, then we say that a is related to b and write, $a\,R\,b$.

If $(a, b) \notin R$, then we say that a is not related to b and write, $a \not R\, b$.

Example Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be a relation in A, given by

$$R = \{(a, b) : a - b = 2\}.$$

Then, $R = \{(3, 1), (4, 2), (5, 3), (6, 4)\}$.

Clearly, $3\,R\,1$, $4\,R\,2$, $5\,R\,3$ and $6\,R\,4$.

But, $1 \not R\, 3$, $2 \not R\, 4$, $5 \not R\, 6$, etc.

DOMAIN AND RANGE OF A RELATION

Let R be a relation in a set A. Then, the set of all first coordinates of elements of R is called the domain of R, written as dom (R) and the set of all second coordinates of R is called the range of R, written as range (R).

$\therefore$ dom $(R) = \{a : (a, b) \in R\}$ and range $(R) = \{b : (a, b) \in R\}$

Example Let $A = \{1, 2, 3, 4, \ldots, 15, 16\}$ and let R be a relation in A, given by

$$R = \{(a, b) : b = a^2\}.$$

Then, $R = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$.

$\therefore$ dom $(R) = \{1, 2, 3, 4\}$ and range $(R) = \{1, 4, 9, 16\}$.

Some Particular Types of Relations

EMPTY RELATION (Or VOID RELATION) *A relation R in a set A is called an empty relation, if no element of A is related to any element of A and we denote such a relation by ϕ.*

Thus, $R = \phi \subseteq A \times A$.

Example Let $A = \{1, 2, 3, 4, 5\}$ and let R be a relation in A, given by

$R = \{(a, b) : a - b = 6\}$.

Clearly, no element $(a, b) \in A \times A$ satisfies the property $a - b = 6$.

$\therefore$ R is an empty relation in A.

UNIVERSAL RELATION *A relation R in a set A is called a universal relation, if each element of A is related to every element of A.*

Thus, $R = (A \times A) \subseteq (A \times A)$ is the universal relation on A.

Example Let $A = \{1, 2, 3\}$. Then,
$R = (A \times A) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
is the universal relation in A.

IDENTITY RELATION *The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.*

Example Let $A = \{1, 2, 3\}$. Then,
$I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

VARIOUS TYPES OF RELATIONS

Let A be a nonempty set. Then, a relation R on A is said to be

(i) **reflexive** if $(a, a) \in R$ for each $a \in A$,
i.e., if $a \, R \, a$ for each $a \in A$.

(ii) **symmetric** if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$,
i.e., if $a \, R \, b \Rightarrow b \, R \, a$ for all $a, b \in A$.

(iii) **transitive** if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$,
i.e., if $a \, R \, b$ and $b \, R \, c \Rightarrow a \, R \, c$.

EQUIVALENCE RELATION *A relation R in a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.*

SOLVED EXAMPLES

EXAMPLE 1 *Let A be the set of all triangles in a plane and let R be a relation in A, defined by $R = \{(\Delta_1, \Delta_2) : \Delta_1 \cong \Delta_2\}$.
Show that R is an equivalence relation in A.*

SOLUTION The given relation satisfies the following properties:

(i) *Reflexivity*

Let Δ be an arbitrary triangle in A. Then,

$\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all values of Δ in A.

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $\Delta_1, \Delta_2 \in A$ such that $(\Delta_1, \Delta_2) \in R$. Then,

$$(\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \cong \Delta_2$$
$$\Rightarrow \Delta_2 \cong \Delta_1$$
$$\Rightarrow (\Delta_2, \Delta_1) \in R.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $\Delta_1, \Delta_2, \Delta_3 \in A$ such that $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$.

Then, $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$

$$\Rightarrow \Delta_1 \cong \Delta_2 \text{ and } \Delta_2 \cong \Delta_3$$
$$\Rightarrow \Delta_1 \cong \Delta_3$$
$$\Rightarrow (\Delta_1, \Delta_3) \in R.$$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

EXAMPLE 2 *Let A be the set of all lines in xy-plane and let R be a relation in A, defined by*

$$R = \{(L_1, L_2) : L_1 \parallel L_2\}.$$

Show that R is an equivalence relation in A.

Find the set of all lines related to the line $y = 3x + 5$.

SOLUTION The given relation satisfies the following properties:

(i) *Reflexivity*

Let L be an arbitrary line in A. Then,

$$L \parallel L \Rightarrow (L, L) \in R \;\; \forall \;\; L \in A.$$

Thus, R is reflexive.

(ii) *Symmetry*

Let $L_1, L_2 \in A$ such that $(L_1, L_2) \in R$. Then,

$$(L_1, L_2) \in R \Rightarrow L_1 \parallel L_2$$
$$\Rightarrow L_2 \parallel L_1$$
$$\Rightarrow (L_2, L_1) \in R.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $L_1, L_2, L_3 \in A$ such that $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$.

Then, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$

$$\Rightarrow L_1 \parallel L_2 \text{ and } L_2 \parallel L_3$$
$$\Rightarrow L_1 \parallel L_3$$
$$\Rightarrow (L_1, L_3) \in R.$$

$\therefore$ R is transitive.

Thus R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The family of lines parallel to the line $y = 3x + 5$ is given by $y = 3x + k$, where k is real.

EXAMPLE 3 *Let Z be the set of all integers and let R be a relation in Z, defined by* $R = \{(a, b) : (a - b) \text{ is even}\}$.
Show that R is an equivalence relation in Z.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*
Let a be an arbitrary element of Z.
Then, $(a - a) = 0$, which is even.
$\therefore (a, a) \in R \ \forall \ a \in Z$.
So, R is reflexive.

(ii) *Symmetry*
Let $a, b \in Z$ such that $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is even
$\Rightarrow -(a - b)$ is even
$\Rightarrow (b - a)$ is even
$\Rightarrow (b, a) \in R$.

$\therefore$ R is symmetric.

(iii) *Transitivity*
Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,
$(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow (a - b)$ is even and $(b - c)$ is even
$\Rightarrow \{(a - b) + (b - c)\}$ is even
$\Rightarrow (a - c)$ is even
$\Rightarrow (a, c) \in R$.
$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.
Hence, R is an equivalence relation in Z.

EXAMPLE 4 *Let A be the set of all lines in a plane and let R be a relation in A defined by*

$$R = \{(L_1, L_2) : L_1 \perp L_2\}.$$

Show that R is symmetric but neither reflexive nor transitive.

SOLUTION Clearly, any line L cannot be perpendicular to itself.
$\therefore (L, L) \notin R$ for any $L \in A$.
So, R is not reflexive.
Again, let $(L_1, L_2) \in R$. Then,

$$(L_1, L_2) \in R \Rightarrow L_1 \perp L_2 \Rightarrow L_2 \perp L_1 \Rightarrow (L_2, L_1) \in R.$$

$\therefore$ R is symmetric.

Now, let $L_1, L_2, L_3 \in A$ such that $L_1 \perp L_2$ and $L_2 \perp L_3$.
Then, clearly L_1 is not perpendicular to L_3.
Thus, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$, but $(L_1, L_3) \notin R$.

$\therefore$ R is not transitive.
Hence, R is symmetric but neither reflexive nor transitive.

EXAMPLE 5 *Let S be the set of all real numbers and let R be a relation in S defined by*
$R = \{(a, b) : (1 + ab) > 0\}$.
Show that R is reflexive and symmetric but not transitive.

SOLUTION Let a be any real number. Then,
$(1 + aa) = (1 + a^2) > 0$ shows that $(a, a) \in R \ \forall \ a \in S$.
$\therefore$ R is reflexive.
Also, $(a, b) \in R \Rightarrow (1 + ab) > 0$
$\Rightarrow (1 + ba) > 0 \quad [\because \ ab = ba]$
$\Rightarrow (b, a) \in R$.
$\therefore$ R is symmetric.
In order to show that R is not transitive, consider $(-1, 0)$ and $(0, 2)$.
Clearly, $(-1, 0) \in R$, since $[1 + (-1) \times 0] > 0$.
And, $(0, 2) \in R$, since $[1 + 0 \times 2] > 0$.
But, $(-1, 2) \notin R$, since $[1 + (-1) \times 2]$ is not greater than 0.
Hence, R is reflexive and symmetric but not transitive.

EXAMPLE 6 *Let S be the set of all real numbers and let R be a relation in S, defined by*
$R = \{(a, b) : a \le b\}$.
Show that R is reflexive and transitive but not symmetric.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*
Let a be an arbitrary real number.
Then, $a \le a \Rightarrow (a, a) \in R$.
Thus, $(a, a) \in R \ \forall \ a \in S$.
$\therefore$ R is reflexive.

(ii) *Transitivity*
Let a, b, c be real numbers such that $(a, b) \in R$ and $(b, c) \in R$.
Then, $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow a \le b$ and $b \le c$
$\Rightarrow a \le c$
$\Rightarrow (a, c) \in R$.
$\therefore$ R is transitive.

(iii) *Nonsymmetry*

Clearly, $(4, 5) \in R$ since $4 \le 5$.

But, $(5, 4) \notin R$ since $5 \le 4$ is not true.

$\therefore$ R is not symmetric.

EXAMPLE 7 *Let S be the set of all real numbers and let R be a relation in S, defined by* $R = \{(a, b) : a \le b^2\}$.

Show that R satisfies none of reflexivity, symmetry and transitivity.

SOLUTION (i) *Nonreflexivity*

Clearly, $\frac{1}{2}$ is a real number and $\frac{1}{2} \le \left(\frac{1}{2}\right)^2$ is not true.

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right) \notin R$.

Hence, R is not reflexive.

(ii) *Nonsymmetry*

Consider the real numbers $\frac{1}{2}$ and 1.

Clearly, $\frac{1}{2} \le 1^2 \Rightarrow \left(\frac{1}{2}, 1\right) \in R$.

But, $1 \le \left(\frac{1}{2}\right)^2$ is not true and so $\left(1, \frac{1}{2}\right) \notin R$.

Thus, $\left(\frac{1}{2}, 1\right) \in R$ but $\left(1, \frac{1}{2}\right) \notin R$.

Hence, R is not symmetric.

(iii) *Nontransitivity*

Consider the real numbers 2, –2 and 1.

Clearly, $2 \le (-2)^2$ and $-2 \le (1)^2$ but $2 \le 1^2$ is not true.

Thus, $(2, -2) \in R$ and $(-2, 1) \in R$, but $(2, 1) \notin R$.

Hence, R is not transitive.

EXAMPLE 8 *Let S be the set of all real numbers and let R be a relation in S, defined by* $R = \{(a, b) : a \le b^3\}$.

Show that R satisfies none of reflexivity, symmetry and transitivity.

SOLUTION (i) *Nonreflexivity*

Clearly, $\frac{1}{2}$ is a real number and $\frac{1}{2} \le \left(\frac{1}{2}\right)^3$ is not true.

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right) \notin R$.

Hence, R is not reflexive.

(ii) *Nonsymmetry*

Take the real numbers $\frac{1}{2}$ and 1.

Clearly, $\frac{1}{2} \le 1^2$ is true and therefore, $\left(\frac{1}{2}, 1\right) \in R$.

But, $1 \le \left(\frac{1}{2}\right)^2$ is not true and so $\left(1, \frac{1}{2}\right) \notin R$.

Hence, R is not symmetric.

(iii) *Nontransitivity*

Consider the real numbers 3, $\frac{3}{2}$ and $\frac{4}{3}$.

Clearly, $3 \le \left(\frac{3}{2}\right)^2$ and $\frac{3}{2} \le \left(\frac{4}{3}\right)^2$ but $3 \le \left(\frac{4}{3}\right)^2$ is not true.

Thus, $\left(3, \frac{3}{2}\right) \in R$ and $\left(\frac{3}{2}, \frac{4}{3}\right) \in R$, but $\left(3, \frac{4}{3}\right) \notin R$.

Hence, R is not transitive.

Thus, R satisfies none of reflexivity, symmetry and transitivity.

EXAMPLE 6 *Let N be the set of all natural numbers and let R be a relation in N, defined by*

$$R = \{(a, b) : a \text{ is a factor of } b\}.$$

Then, show that R is reflexive and transitive but not symmetric.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary element of N.

Then, clearly, a is a factor of a.

$\therefore \ (a, a) \in R \ \forall \ a \in N$.

So, R is reflexive.

(ii) *Transitivity*

Let $a, b, c \in N$ such that $(a, b) \in R$ and $(b, c) \in R$.

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ (a is a factor of b) and (b is a factor of c)

$\Rightarrow$ $b = ad$ and $c = be$ for some $d, e \in N$

$\Rightarrow$ $c = (ad)\,e = a(de)$ [by associative law]

$\Rightarrow$ a is a factor of c

$\Rightarrow$ $(a, c) \in R$.

$\therefore$ $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

Hence, R is transitive.

(iii) *Nonsymmetry*

Clearly, 2 and 6 are natural numbers and 2 is a factor of 6.

$\therefore \ (2, 6) \in R$.

But, 6 is not a factor of 2.

$\therefore$ $(6, 2) \notin R$.

Thus, $(2, 6) \in R$ and $(6, 2) \notin R$.

Hence, R is not symmetric.

EXAMPLE 10 *Let N be the set of all natural numbers and let R be a relation in N, defined by*

$R = \{(a, b) : a \text{ is a multiple of } b\}$.

Show that R is reflexive and transitive but not symmetric.

SOLUTION Here R satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary element of N.

Then, $a = (a \times 1)$ shows that a is a multiple of a.

$\therefore$ $(a, a) \in R \ \forall \ a \in N$.

So, R is reflexive.

(ii) *Transitivity*

Let $a, b, c \in N$ such that $(a, b) \in R$ and $(b, c) \in R$.

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ (a is a multiple of b) and (b is a multiple of c)

$\Rightarrow$ $a = bd$ and $b = ce$ for some $d \in N$ and $e \in N$

$\Rightarrow$ $a = (ce)d$

$\Rightarrow$ $a = c(ed)$

$\Rightarrow$ a is a multiple of c

$\Rightarrow$ $(a, c) \in R$.

$\therefore$ $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

Hence, R is transitive.

(iii) *Nonsymmetry*

Clearly, 6 and 2 are natural numbers and 6 is a multiple of 2.

$\therefore$ $(6, 2) \in R$.

But, 2 is not a multiple of 6.

$\therefore$ $(2, 6) \notin R$.

Thus, $(6, 2) \in R$ and $(2, 6) \notin R$.

Hence, R is not symmetric.

EXAMPLE 11 *Let X be a nonempty set and let S be the collection of all subsets of X. Let R be a relation in S, defined by*

$R = \{(A, B) : A \subset B\}$.

Show that R is transitive but neither reflexive nor symmetric.

SOLUTION Clearly, R satisfies the following properties:

(i) *Transitivity*

Let $A, B, C \in S$ such that $(A, B) \in R$ and $(B, C) \in R$.

Now, $(A, B) \in R$ and $(B, C) \in R$

$\Rightarrow A \subset B$ and $B \subset C$

$\Rightarrow A \subset C$

$\Rightarrow (A, C) \in R.$

$\therefore$ R is transitive.

(ii) *Nonreflexivity*

Let A be any set in S.

Then, $A \not\subset A$ shows that $(A, A) \notin R$.

$\therefore$ R is not reflexive.

(iii) *Nonsymmetry*

Now $(A, B) \in R \Rightarrow A \subset B$

$\Rightarrow B \not\subset A$

$\Rightarrow (B, A) \notin R.$

$\therefore$ R is not symmetric.

Hence, R is transitive but neither reflexive nor symmetric.

EXAMPLE 12 *Give an example of a relation which is*

(i) reflexive and transitive but not symmetric;

(ii) symmetric and transitive but not reflexive;

(iii) reflexive and symmetric but not transitive;

(iv) symmetric but neither reflexive nor transitive;

(v) transitive but neither reflexive nor symmetric.

SOLUTION Let $A = \{1, 2, 3\}$.

Then, it is easy to verify that the relation

(i) $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$

is reflexive and transitive.

R_1 is not symmetric, since

$(1, 2) \in R$ and $(2, 1) \notin R$.

(ii) $R_2 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$

is symmetric and transitive.

But, R_2 is not reflexive, since $(3, 3) \notin R_2$.

(iii) $R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$

is reflexive and symmetric.

But, R_3 is not transitive, since

$(1, 2) \in R_3$, $(2, 3) \in R_3$ but $(1, 3) \notin R_3$.

(iv) $R_4 = \{(2, 2), (3, 3), (1, 2), (2, 1)\}$

is symmetric.

But, R_4 is not reflexive since $(1, 1) \notin R_4$.

Also, R_4 is not transitive, as

$(1, 2) \in R_4$ and $(2, 1) \in R_4$ but $(1, 1) \notin R_4$.

(v) $R_5 = \{(2, 2), (3, 3), (1, 2)\}$

is transitive.

But, R_5 is not reflexive, since $(1, 1) \notin R$.

And, R_5 is not symmetric as $(1, 2) \in R_5$ but $(2, 1) \notin R_5$.

नोट :-> यह transitive इसलिए है क्योंकि (a, b), (b, c) के form में है ही नहीं इसलिए इसे (a, c) की आवश्यकता ही नहीं है

EXAMPLE 13 *Let N be the set of all natural numbers and let R be a relation on N × N, defined by*

$(a, b) R (c, d) \Leftrightarrow ad = bc.$

Show that R is an equivalence relation.

SOLUTION Here R satisfies the following properties:

(i) *Reflexivity*

Let $(a, b) \in R$. Then,

$(a, b) R (a, b)$, since $ab = ba$ [by commutative law of multiplication on N].

Thus, $(a, b) R (a, b) \; \forall \; (a, b) \in R$.

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $(a, b) R (c, d)$. Then,

$(a, b) R (c, d) \Rightarrow ad = bc$

$\Rightarrow bc = ad$

$\Rightarrow cb = da$ [by commutativity of multiplication on N]

$\Rightarrow (c, d) R (a, b)$.

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b) R (c, d)$ and $(c, d) R (e, f)$. Then,

$ad = bc$ and $cf = de$

$\Rightarrow adcf = bcde$

$\Rightarrow (af)(cd) = (be)(cd)$

$\Rightarrow af = be$ [by cancellation law]

$\Rightarrow (a, b) R (e, f)$.

$\therefore$ $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow (a, b) R (e, f)$.

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

EXAMPLE 14 *If R_1 and R_2 be two equivalence relations on a set A, prove that $R_1 \cap R_2$ is also an equivalence relation on A.*

SOLUTION Let R_1 and R_2 be two equivalence relations on a set A.

Then, $R_1 \subseteq A \times A$, $R_2 \subseteq A \times A \Rightarrow (R_1 \cap R_2) \subseteq A \times A$.

So, $(R_1 \cap R_2)$ is a relation on A.

This relation on A satisfies the following properties.

(i) *Reflexivity*

R_1 is reflexive and R_2 is reflexive

$\Rightarrow (a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$

$\Rightarrow (a, a) \in R_1 \cap R_2$ for all $a \in A$

$\Rightarrow R_1 \cap R_2$ is reflexive.

(ii) *Symmetry*

Let (a, b) be an arbitrary element of $R_1 \cap R_2$. Then,

$(a, b) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2$

[$\because$ R_1 is symmetric and R_2 is symmetric]

$\Rightarrow (b, a) \in R_1 \cap R_2$.

This shows that $R_1 \cap R_2$ is symmetric.

(iii) *Transitivity*

$(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$, $(a, b) \in R_2$, and $(b, c) \in R_1$, $(b, c) \in R_2$

$\Rightarrow \{(a, b) \in R_1, (b, c) \in R_1\}$, and $\{(a, b) \in R_2, (b, c) \in R_2\}$

$\Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2$

[$\because$ R_1 is transitive and R_2 is transitive]

$\Rightarrow (a, c) \in R_1 \cap R_2$.

This shows that $(R_1 \cap R_2)$ is transitive.

Thus, $R_1 \cap R_2$ is reflexive, symmetric and transitive.

Hence, $R_1 \cap R_2$ is an equivalence relation.

EXAMPLE 15 *Give an example to show that the union of two equivalence relations on a set A need not be an equivalence relation on A.*

SOLUTION Let R_1 and R_2 be two relations on a set $A = \{1, 2, 3\}$, given by

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

and $R_2 = \{(1, 1,), (2, 2), (3, 3), (1, 3), (3, 1)\}$.

Then, it is easy to verify that each one of R_1 and R_2 is an equivalence relation.

But, $R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$

is not transitive, as

$(3, 1) \in R_1 \cup R_2$ and $(1, 2) \in R_1 \cup R_2$ but $(3, 2) \notin R_1 \cup R_2$.

Hence, $(R_1 \cup R_2)$ is not an equivalence relation.

EQUIVALENCE CLASSES *Let R be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called the equivalence class determined by a and it is denoted by $[a]$.*

Thus, $[a] = \{b \in A : (a, b) \in R\}$.

Two equivalence classes are either disjoint or identical.

An Important Result *An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.*

EXAMPLE 16 *On the set Z of all integers, consider the relation*
$R = \{(a, b) : (a - b) \text{ is divisible by } 3\}$.
Show that R is an equivalence relation on Z.
Also find the partitioning of Z into mutually disjoint equivalence classes.

SOLUTION The relation R on Z satisfies the following properties:

(i) *Reflexivity*
Let $a \in Z$.
Then, $(a - a) = 0$, which is divisible by 3
$\therefore\ a R a \ \forall\ a \in Z$.
So, R is reflexive.

(ii) *Symmetry*
Let $a, b \in Z$ such that $a R b$. Then,
$a R b \Rightarrow a - b$ is divisible by 3
$\Rightarrow -(a - b)$ is divisible by 3
$\Rightarrow (b - a)$ is divisible by 3
$\Rightarrow b R a$.
$\therefore\ a R b \Rightarrow b R a \ \forall\ a, b \in Z$.
So, R is symmetric.

(iii) *Transitivity*
Let $a, b, c \in Z$ such that $a R b$ and $b R c$. Then,
$a R b, b R c \Rightarrow (a - b)$ is divisible by 3
and $(b - c)$ is divisible by 3
$\Rightarrow [(a - b) + (b - c)]$ is divisible by 3
$\Rightarrow (a - c)$ is divisible by 3.
$\Rightarrow a R c \ \forall\ a, b, c \in Z$.

Thus, $a R b, b R c \Rightarrow a R c \ \forall\ a, b, c \in Z$.
$\therefore$ R is an equivalence relation on Z.

Read again

Now, let us consider [0], [1] and [2].
We have:
$[0] = \{x \in Z : x R 0\}$
$= \{x \in Z : (x - 0) \text{ is divisible by } 3\}$
$= \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$.
$\therefore\ [0] = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$.

Similarly, $[1] = \{x \in Z : x R 1\}$
$= \{x \in Z : (x - 1) \text{ is divisible by } 3\}$
$= \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$.
$\therefore\ [1] = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$.

And, $[2] = \{x \in Z : x R 2\}$
$= \{x \in Z : (x - 2) \text{ is divisible by } 3\}$
$= \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$.
$\therefore\ [2] = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$.

Clearly, [0], [1] and [2] are mutually disjoint and $Z = [0] \cup [1] \cup [2]$.

EXAMPLE 17 *Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and let R be a relation on A, defined by*
$R = \{(a, b)$: both a and b are either odd or even$\}$.
Prove that R is an equivalence relation.
Let $B = \{1, 3, 5, 7\}$ and $C = \{2, 4, 6\}$.
Show that

(a) all elements of B are related to each other;
(b) all elements of C are related to each other;
(c) no element of B is related to any element of C.

SOLUTION The given relation satisfies the following properties:

(i) *Reflexivity*

Let $a \in A$.

Then, it is clear that a and a are both odd or both even.

$\therefore \ (a, a) \in R \ \forall \ a \in A$.

So, R is reflexive.

(ii) *Symmetry*

Let $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow$ both a and b are either odd or even
$\Rightarrow$ both b and a are either odd or even
$\Rightarrow (b, a) \in R$.

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ {both a and b are either odd or even} and {both b and c are either odd or even}
$\Rightarrow$ both a and c are either odd or even
$\Rightarrow (a, c) \in R$.

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

(a) If we pick up any two elements of B, then both being odd, they are related to each other.

(b) If we pick up any two elements of C, then both being even, they are related to each other.

(c) If we pick up one element of B and one element of C, then one is even while the other is odd.
So, they are not related to each other.

EXERCISE 1A

1. Define a relation on a set. What do you mean by the domain and range of a relation?

2. Find the domain and range of the relation
$R = \{(-1, 1), (1, 1), (2, 4), (-2, 4)\}$.

3. Let $A = \{1, 2, 3, 4, 6\}$ and let R be a relation on A, defined by $R = \{(a, b) : a \in A,\ b \in A \text{ and } a \text{ divides } b\}$.
Find (i) R, (ii) dom (R) and (iii) range (R).

4. Let $R = \{(a, a^3) : a \text{ is a prime number less than } 10\}$.
Find (i) R, (ii) dom (R) and (iii) range (R).

5. List the elements of each of the following relations. Find the domain and range in each case.
   (i) $R_1 = \left\{\left(a, \frac{1}{a}\right) : a \in N \text{ and } 1 \le a \le 5\right\}$.
   (ii) $R_2 = \{(a, b) : a \in N,\ b \in N \text{ and } a + 3b = 12\}$
   (iii) $R_3 = \{(a, b) : b = |a - 1|,\ a \in Z \text{ and } |a| \le 3\}$

6. Let A be the set of all triangles in a plane. Show that the relation $R = \{(\Delta_1, \Delta_2) : \Delta_1 \sim \Delta_2\}$ is an equivalence relation on A.

7. Let $R = \{(a, b) : a,\ b \in Z \text{ and } (a + b) \text{ is even}\}$.
Show that R is an equivalence relation on Z.

8. Let $R = \{(a, b) : a,\ b \in Z \text{ and } (a - b) \text{ is divisible by } 5\}$.
Show that R is an equivalence relation on Z.

9. Let S be the set of all real numbers and let $R = \{(a, b) : a,\ b \in S \text{ and } a = \pm b\}$.
Show that R is an equivalence relation on S.

10. Let S be the set of all points in a plane and let R be a relation in S defined by $R = \{(A, B) : d(A, B) < 2 \text{ units}\}$, where $d(A, B)$ is the distance between the points A and B.
Show that R is reflexive and symmetric but not transitive.

11. Show that the relation R defined on $A = \{x \in Z : 0 \le x \le 12\}$, given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation. Find the set of elements related to 1.

12. Let $R = \{(a, b) : a = b^2\}$ for all $a,\ b \in N$.
Show that R satisfies none of reflexivity, symmetry and transitivity.

13. Let A be the set of all points in a plane and let O be the origin. Show that the relation R, defined by
$$R = \{(P, Q) : OP = OQ\}$$
is an equivalence relation.

14. Show that the relation $R = \{(a, b) : a > b\}$ on N is transitive but neither reflexive nor symmetric.

15. Let S be the set of all real numbers. Show that the relation $R = \{(a, b) : a^2 + b^2 = 1\}$ is symmetric but neither reflexive nor transitive.

[CBSE 2008]

16. Show that the relation R on $N \times N$, defined by
$(a, b) R (c, d) \Leftrightarrow a + d = b + c$
is an equivalent relation.

17. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 2)\}$. Show that R is reflexive and transitive but not symmetric.

18. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$. Show that R is reflexive but neither symmetric nor transitive.

ANSWERS (EXERCISE 1A)

2. dom $(R) = \{-1, 1, -2, 2\}$, range $(R) = \{1, 4\}$
3. (i) $R = \{(1, 2), (1, 3), (1, 4), (1, 6), (2, 4), (2, 6), (3, 6)\}$
   (ii) dom $(R) = \{1, 2, 3\}$
   (iii) range $(R) = \{2, 3, 4, 6\}$
4. (i) $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$
   (ii) dom $R = \{2, 3, 5, 7\}$
   (iii) range $(R) = \{8, 27, 125, 343\}$
5. (i) $R_1 = \left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$
   dom $(R_1) = \{1, 2, 3, 4, 5\}$, range $(R_1) = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$
   (ii) $R_2 = \{(3, 3), (6, 2), (9, 1)\}$, dom $(R_2) = \{3, 6, 9\}$, range $(R_2) = \{3, 2, 1\}$
   (iii) $R_3 = \{(-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1), (3, 2)\}$
   dom $(R_3) = \{-3, -2, -1, 0, 1, 2, 3\}$, range $(R_3) = \{4, 3, 2, 1, 0\}$.
11. $\{1, 3, 5, 7, 9, 11\}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 1A)

5. $a = -3, -2, -1, 0, 1, 2, 3$.
7. We shall prove transitivity.
   $(a, b) \in R, (b, c) \in R$
   $\Rightarrow (a + b)$ is even, $(b + c)$ is even
   $\Rightarrow (a + c) = \{(a + c) + 2b\} - 2b$
   $= [(a + b) + (b + c) - 2b]$, which is even
   [$\because$ $(a + b)$ is even, $(b + c)$ is even, $(-2b)$ is even]
   $\Rightarrow (a, c) \in R$.
10. (i) Clearly, $d(A, A) = 0 < 2 \Rightarrow (A, A) \in R$.
   (ii) $(A, B) \in R \Rightarrow d(A, B) < 2$
   $\Rightarrow d(B, A) < 2$ [$\because d(B, A) = d(A, B)$]
   $\Rightarrow (B, A) \in R$.
   (iii) Consider the points $A(0, 0)$, $B(1.5, 0)$, $C(3, 0)$.

Then, $d(A, B) = 1.5$, $d(B, C) = 1.5$ and $d(A, C) = 3$.

$\therefore$ $d(A, C) \le d(A, B) + d(B, C)$ is not true.

Hence, R is reflexive and symmetric but not transitive.

11. (i) Let $a \in A$. Then, $|a - a| = 0$, which is even.

$\therefore$ $aRa \;\forall\; a \in A$.

(ii) $(a, b) \in R \Rightarrow |a - b|$ is even

$\Rightarrow |-(a - b)|$ is even $\Rightarrow |b - a|$ is even

$\Rightarrow (b, a) \in R$.

(iii) $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a - b) = \pm 2k_1$ and $(b - c) = \pm 2k_2$ for some $k_1, k_2 \in N$

$\Rightarrow [(a - b) + (b - c)] = \pm 2(k_1 \pm k_2)$

$\Rightarrow (a - c) = \pm 2k$ for some $k \in N$

$\Rightarrow (a, c) \in R$.

Set of elements related to 1

$= \{b \in A : |1 - b|$ is even$\}$

$= \{1, 3, 5, 7, 9, 11\}$.

12. (i) $2 \neq 2^2 \Rightarrow 2$ is not related to 2.

(ii) $4 = 2^2 \Rightarrow 4 R 2$. But $2 \neq 4^2$. So, $2 \not R 4$.

(iii) $16 R 4$, $4 R 2$. But 16 is not related to 2.

13. (i) $OP = OP$.

(ii) $OP = OQ \Rightarrow OQ = OP$.

(iii) $OP = OQ$ and $OQ = OR$

$\Rightarrow OP = OR$.

16. (i) $a + b = b + a \Rightarrow (a, b) R (a, b)$.

(ii) $(a, b) R (c, d) \Rightarrow a + d = b + c$

$\Rightarrow b + c = a + d$

$\Rightarrow c + b = d + a$

$\Rightarrow (c, d) R (a, b)$.

(iii) $(a, b) R (c, d)$ and $(c, d) R (e, f)$

$\Rightarrow (a + d) = (b + c)$ and $(c + f) = (d + e)$

$\Rightarrow a + d + c + f = b + c + d + e$

$\Rightarrow a + f = b + e$

$\Rightarrow (a, b) R (e, f)$.

EXERCISE 1B

Very-Short-Answer Questions

1. Show that the relation R in the set Z of all integers, defined by

$$R = \{(a, b) : (a - b) \text{ is an integer}\}$$

is (i) reflexive (ii) symmetric (iii) transitive.

2. On the set S of all real numbers, define a relation $R = \{(a, b) : a \le b\}$. Show that R is (i) reflexive (ii) transitive (iii) not symmetric.

3. On the set S of all real numbers, define a relation $R = \{(a, b) : a \le b^3\}$. Show that R is (i) not reflexive (ii) not symmetric (iii) not transitive.

4. On the set S of all real numbers, define a relation $R = \{(a, b) : a \le b^2\}$. Show that R is (i) not reflexive (ii) not symmetric (iii) not transitive.
5. On the set S of all real numbers, define a relation $R = \{(a, b) : 1 + ab > 0\}$. Show that R is (i) reflexive (ii) symmetric (iii) not transitive.
6. Let S be the set of all sets and let $R = \{(A, B) : A \subset B\}$, i.e., A is a proper subset of B. Show that R is (i) transitive (ii) not reflexive (iii) not symmetric.
7. Let $A = \{1, 2, 3, 4, 5, 6\}$. Consider a relation R on A, defined by $R = \{(a, b) : b = a + 1\}$. Show that R is (i) not reflexive (ii) not symmetric (iii) not transitive.
8. Let $A = \{x \in Z : 0 \le x \le 12\}$. Show that the relation $R = \{(a, b) : |a - b|$ is a multiple of 4$\}$ is (i) reflextive (ii) symmetric (iii) transitive.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 1B)

1. Let $a, b, c \in Z$.
   (i) Since $a - a = 0$, which is an integer;
   $\therefore (a, a) \in R \;\forall\; a \in Z$.
   $\therefore R$ is reflexive.
   (ii) $a R b \Rightarrow (a - b)$ is an integer
   $\Rightarrow -(a - b)$ is an integer $\Rightarrow (b - a)$ is an integer $\Rightarrow b R a$.
   $\therefore R$ is symmetric.
   (iii) $a R b, b R c \Rightarrow (a - b)$ is an integer and $(b - c)$ is an integer
   $\Rightarrow (a - b) + (b - c)$ is an integer
   $\Rightarrow (a - c)$ is an integer $\Rightarrow a R c$.
   $\therefore R$ is transitive.
2. Let a, b, c be arbitrary real numbers. Then
   (i) $a \le a \Rightarrow a R a$. So, R is reflexive.
   (ii) $a R b, b R c \Rightarrow a \le b$ and $b \le c \Rightarrow a \le c \Rightarrow a R c$.
   $\therefore R$ is transitive.
   (iii) $-2 \le 1$ shows that -2 is related to 1.
   But, 1 is greater than -2. So, 1 is not related to -2.
   $\therefore R$ is not symmetric.
3. (i) Clearly, $\frac{1}{2} > \left(\frac{1}{2}\right)^3$. So, $\frac{1}{2}$ is not related to $\frac{1}{2}$.
   Hence, R is not reflexive.
   (ii) $1 \le 2^3$ means 1 is related to 2.
   But, $2 \le 1^3$ is not true. So, 2 is not related to 1.
   So, R is not symmetric.
   (iii) $30 \le 4^3$ and $4 \le 3^3$. But, $30 \le 3^3$ is not true.
   Thus $30 R 4$ and $4 R 3$. But, 30 is not related to 3.
   $\therefore R$ is not transitive.
4. (i) $\frac{1}{2} \le \left(\frac{1}{2}\right)^2$ is not true. So, R is not reflexive.

(ii) $1 \le 2^2$ is clearly true.
But, $2 \le 1^2$ is not true.
$\therefore$ 1 R 2. But, 2 is not related to 1.
(iii) $10 \le 4^2$ and $4 \le 3^2$. But, $10 \le 3^2$ is not true.
Thus, 10 R 4 and 4 R 3. But, 10 is not related to 3.

5. (i) Let a be an arbitrary real number. Then
$(1 + a \cdot a) = (1 + a^2) > 0$. So, $a\,R\,a$ for all $a \in S$.
$\therefore$ R is reflexive.
(ii) $a\,R\,b \Rightarrow (1 + ab) > 0$
$\Rightarrow (1 + ba) > 0 \Rightarrow b\,R\,a$.
$\therefore$ R is symmetric.
(iii) Let $a = 1, b = \frac{1}{2}$ and $c = -1$. Then
$a\,R\,b$, since $\left(1 + 1 \times \frac{1}{2}\right) > 0$
$b\,R\,c$, since $\left[1 + \frac{1}{2} \times (-1)\right] = \frac{1}{2} > 0$.
But $[1 + 1 \times (-1)]$ is not greater than 0.
So, a is not related to c.
$\therefore$ R is not transitive.

6. (i) $A\,R\,B, B\,R\,C \Rightarrow A \subset B$ and $B \subset C$
$\Rightarrow A \subset C \Rightarrow A\,R\,C$.
$\therefore$ R is transitive.
(ii) $A \subset A$ is not true. So, R is not reflexive.
(iii) $\{1, 2\} \subset \{1, 2, 3\}$. But, $\{1, 2, 3\} \subset \{1, 2\}$ is not true.
$\therefore$ R is not symmetric.

7. Clearly, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$.
(i) Since $(1, 1) \notin R$, so R is not reflexive.
(ii) Clearly, $(1, 2) \in R$. But, $(2, 1) \notin R$.
$\therefore$ R is not symmetric.
(iii) Clearly, $(1, 2) \in R$ and $(2, 3) \in R$. But, $(1, 3) \notin R$.
$\therefore$ (1 R 2 and 2 R 3). But, 1 is not related to 3.
$\therefore$ R is not transitive.

8. (i) $|a - a| = 0$, which is a multiple of 4. So, $a\,R\,a$. So, R is reflexive.
(ii) $a\,R\,b \Rightarrow |a - b|$ is a multiple of 4
$\Rightarrow |-(a - b)|$ is a multiple of 4 $\Rightarrow |b - a|$ is a multiple of 4
$\Rightarrow b\,R\,a$.
$\therefore$ R is symmetric.
(iii) $a\,R\,b, b\,R\,c \Rightarrow |a - b|$ is a multiple of 4 and $|b - c|$ is a multiple of 4.
Let $|a - b| = 4k_1$ and $|b - c| = 4k_2$. Then,
$|a - c| = |(a - b) - (b - c)| = |4k_1 - 4k_2| = |4(k_1 - k_2)| = 4|k_1 - k_2|$, which is a multiple of 4.
$\therefore a\,R\,b, b\,R\,c \Rightarrow a\,R\,c$. So, R is transitive.

2. FUNCTIONS

FUNCTION *Let A and B be two nonempty sets. Then, a rule f which associates to each element $x \in A$, a unique element, denoted by $f(x)$ of B, is called a function from A to B and we write,*

$$f : A \to B.$$

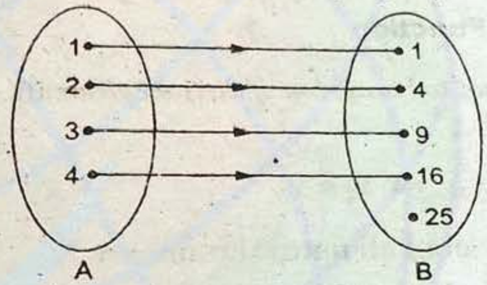
*$f(x)$ is called the **image** of x, while x is called the **pre-image** of $f(x)$.*

Domain, Codomain and Range of a Function

Let $f : A \to B$. Then, A is called the *domain* of f and B is called the *codomain* of f.

And, $f(A) = \{f(x) : x \in A\}$ is called the *range* of f.

Example 1 Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$.



Consider the rule $f : A \to B : f(x) = x^2 \ \forall \ x \in A$.

Then, each element in A has its unique image in B.

So, f is a function from A to B.

$$f(1) = 1^2 = 1,\ f(2) = 2^2 = 4,\ f(3) = 3^2 = 9,\ f(4) = 4^2 = 16.$$

Dom $(f) = \{1, 2, 3, 4\} = A$, codomain $(f) = \{1, 4, 9, 16, 25\} = B$

and range $(f) = \{1, 4, 9, 16\}$.

Clearly, $25 \in B$ does not have its pre-image in A.

Example 2 Let N be the set of all natural numbers.

Let $f : N \to N : f(x) = 2x \ \forall \ x \in N$.

Then, every element in N has its unique image in N.

So, f is a function from N to N.

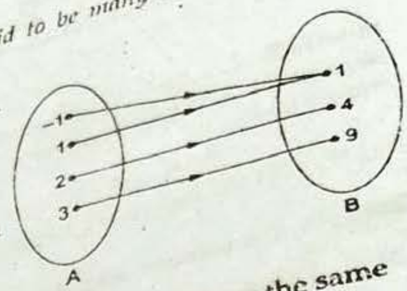
Clearly, $f(1) = 2,\ f(2) = 4,\ f(3) = 6$ ..., and so on.

Dom $(f) = N$, codomain $(f) = N$,

range $(f) = \{2, 4, 6, 8, 10 \ldots\}$.

Various Types of Functions

MANY-ONE FUNCTION *A function $f : A \to B$ is said to be many-one if two or more than two elements in A have the same image in B.*



Example Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$.
Let $f : A \to B : f(x) = x^2 \;\forall\; x \in A$.
Then, each element in A has a unique image under f in B.
$\therefore$ f is a function from A to B such that
$f(-1) = (-1)^2 = 1$; $f(1) = 1^2 = 1$;
$f(2) = 2^2 = 4$ and $f(3) = 3^2 = 9$.
Clearly, two different elements, namely -1 and 1, have the same image $1 \in B$.
Hence, f is many-one.

One-One or Injective Function

A function $f : A \to B$ is said to be one-one if distinct elements in A have distinct images in B.

f is one-one when $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Example Let N be the set of all natural numbers.
Let $f : N \to N : f(x) = 2x \;\forall\; x \in N$.
Then, $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2$
$\Rightarrow x_1 = x_2$.

$\therefore$ f is one-one.

Onto or Surjective Function

A function $f : A \to B$ is said to be onto if every element in B has at least one pre-image in A.

Thus, if f is onto, then for each $y \in B \;\exists$ at least one element $x \in A$ such that $y = f(x)$.

Also, f is onto $\Leftrightarrow$ range $(f) = B$.

Example Let N be the set of all natural numbers and let E be the set of all even natural numbers.

Let $f : N \to E : f(x) = 2x \;\forall\; x \in N$.

Then, $y = 2x \Rightarrow x = \frac{1}{2}y$.

Thus, for each $y \in E$ there exists $\frac{1}{2}y \in N$ such that

$$f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y.$$

$\therefore$ f is onto.

INTO FUNCTION *A function $f : A \to B$ is said to be into if there exists even a single element in B having no pre-image in A.*

Clearly, f is into $\Leftrightarrow$ range $(f) \subset B$.

Example Let $A = \{2, 3, 5, 7\}$ and $B = \{0, 1, 3, 5, 7\}$.

Let $f : A \to B : f(x) = (x - 2)$. Then,

$f(2) = (2 - 2) = 0$, $f(3) = (3 - 2) = 1$, $f(5) = (5 - 2) = 3$ and

$f(7) = (7 - 2) = 5$.

Thus, every element in A has a unique image in B.

Now, $\exists\, 7 \in B$ having no pre-image in A.

$\therefore$ f is into.

Note that range $(f) = \{0, 1, 3, 5\} \subset B$.

Bijective Function

A one-one onto function is said to be bijective or a one-to-one correspondence.

CONSTANT FUNCTION *A function $f : A \to B$ is called a constant function if every element of A has the same image in B.*

Example Let $A = \{1, 2, 3\}$ and $B = \{5, 7, 9\}$.

Let $f : A \to B : f(x) = 5$ for all $x \in A$.

Clearly, every element in A has the same image.

So, f is a constant function.

REMARK The range of a constant function is a singleton set.

IDENTITY FUNCTION *The function $I_A : A \to A : I_A(x) = x \ \forall\ x \in A$ is called an* [illegible]

EQUAL FUNCTIONS *Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)\ \forall\ x \in D$.*

SOLVED EXAMPLES

EXAMPLE 1 *Let $f : N \to N : f(x) = 2x$ for all $x \in N$.*
Show that f is one-one and into.

SOLUTION We have

$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

$\therefore$ f is one-one.

Let $y = 2x$. Then, $x = \frac{y}{2}$.

If we put $y = 3$, then $x = \frac{3}{2} \notin N$.

Thus, $3 \in N$ has no pre-image in N.

$\therefore$ f is into.

Hence, f is one-one and into.

EXAMPLE 2 *Show that the function $f : R \to R : f(x) = x^2$ is neither one-one nor onto.*

SOLUTION We have $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider -1 in the codomain R, then it is clear that there is no element in R whose image is -1.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 3 *Show that the function $f : R \to R : f(x) = |x|$ is neither one-one nor onto.*

SOLUTION We have $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider -1 in the codomain R, then it is clear that there is no real number x whose modulus is -1.

Thus, $-1 \in R$ has no pre-image in R.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 4 *For any real number x, we define*

$[x]$ = greatest integer less than or equal to x.

Prove that the greatest integer function $f : R \to R : f(x) = [x]$ is neither one-one nor onto.

SOLUTION Clearly, $[1.2] = 1$ and $[1.3] = 1$.

$\therefore$ $f(1.2) = 1$ and $f(1.3) = 1$.

Thus, two different real numbers have the same image.

$\therefore$ f is not one-one.

Clearly, there is no real number x such that

$f(x) = [x] = 1.1$.

So, f is not onto.

Hence, f is neither one-one nor onto.

Required probability = P(3 heads or 4 heads or 5 heads or 6 heads)

$$= {}^6C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^3 + {}^6C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 + {}^6C_5 \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right) + {}^6C_6 \cdot \left(\frac{1}{2}\right)^6$$

$$= \left(20 \times \frac{1}{64} + 15 \times \frac{1}{64} + 6 \times \frac{1}{64} + \frac{1}{64}\right) = \frac{42}{64} = \frac{21}{32}.$$

22. In a single throw, we have $P(T) = \frac{1}{2}$ and $P(\text{not } T) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 5$.

Required probability = $P(X = 1)$ or $P(X = 3)$ or $P(X = 5)$

$$= P(X = 1) + P(X = 3) + P(X = 5)$$

$$= {}^5C_1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^4 + {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5$$

$$= \left(\frac{5}{32} + \frac{10}{32} + \frac{1}{32}\right) = \frac{16}{32} = \frac{1}{2}.$$

23. In a single throw, we have $P(H) = \frac{1}{2}$ and $P(\text{not } H) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 5$.

Required probability = $P(X = 0)$ or $P(X = 2)$ or $P(X = 4)$

$$= P(X = 0) + P(X = 2) + P(X = 4)$$

$$= {}^5C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^5 + {}^5C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 + {}^5C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^1$$

$$= \left(\frac{1}{32} + \frac{10}{32} + \frac{5}{32}\right) = \frac{16}{32} = \frac{1}{2}.$$

24. In a single throw, we have $P(H) = \frac{1}{2}$ and $P(\text{not } H) = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 8$.

Required probability = P(6 heads or 7 heads or 8 heads)

$$= P(6 \text{ heads}) + P(7 \text{ heads}) + P(8 \text{ heads})$$

$$= {}^8C_6 \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^2 + {}^8C_7 \cdot \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^1 + {}^8C_8 \cdot \left(\frac{1}{2}\right)^8$$

$$= \left(\frac{28}{256} + \frac{8}{256} + \frac{1}{256}\right) = \frac{37}{256}.$$

25. In a single throw of a die, $P(\text{getting an odd number}) = \frac{3}{6} = \frac{1}{2}$.

$\therefore \quad p = \frac{1}{2}, q = (1 - p) = \frac{1}{2}$ and $n = 5$.

Required probability = P(4 successes or 5 successes)

$$= {}^5C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^1 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5 = \left(\frac{5}{32} + \frac{1}{32}\right) = \frac{6}{32} = \frac{3}{16}.$$