Resonance
Educating for better tomorrow

MATHEMATICS

Ellipse

Target : AIEEE

Class : XI

Ellipse

Our notion of symmetry is derived from the human face. Hence, we demand symmetry horizontally and in breadth only, not vertically nor in depth .... Pascal, Blaise

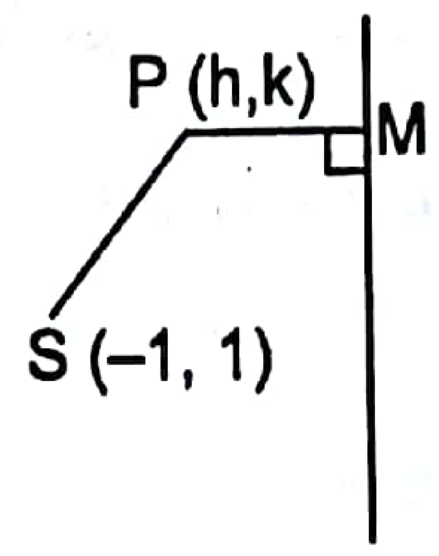
In this chapter we are going to discuss in detail the nature of path in which on planets move around the sun. They follow on elliptical path with the sun at one of its foci. Let us look at the definition of ellipse.

Definitions : -

It is locus of a point which moves in such a way that the ratio of its distance from a fixed point and a fixed line (not passes through fixed point and all points and line lies in same plane) is constant (e), which is less than one.
The fixed point is called - **focus**
The fixed line is called **-directrix.**
The constant ratio is called - **eccentricity**, it is denoted by **'e'**.

Example # 1 : Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $\frac{1}{2}$.

Solution : Let $P \equiv (h, k)$ be moving point,



$$e = \frac{PS}{PM} = \frac{1}{2}$$

$$\Rightarrow \quad (h + 1)^2 + (k - 1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$$

$\Rightarrow$ locus of P(h, k) is

$$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$$

$$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0.$$

Note : The general equation of a conic with focus (p, q) & directrix $\ell x + my + n = 0$ is:

$$(\ell^2 + m^2)[(x - p)^2 + (y - q)^2] = e^2(\ell x + my + n)^2$$

$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent ellipse if $0 < e < 1$; $\Delta \neq 0$, $h^2 < ab$

Self Practice Problem :

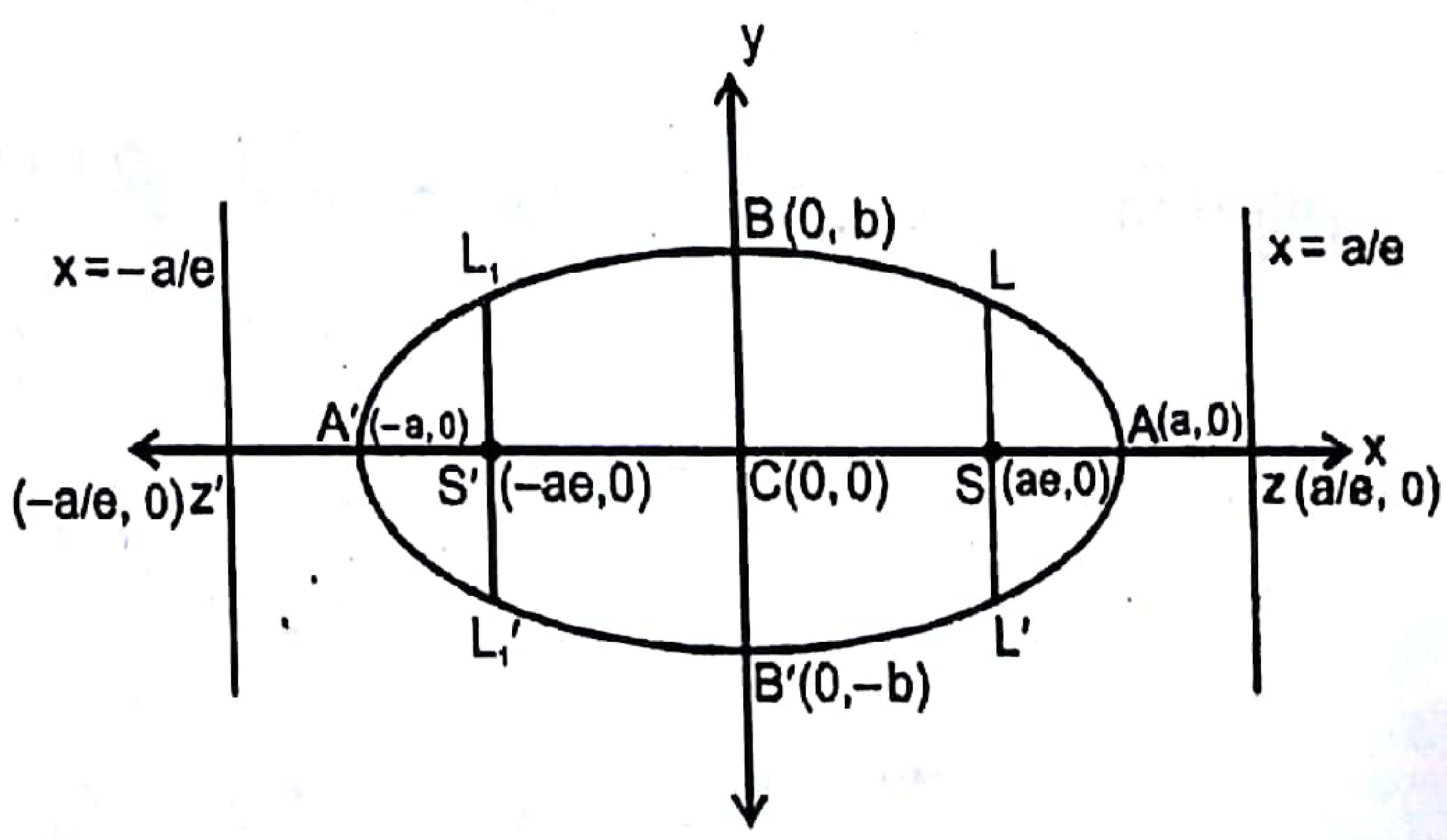
(1) Find the equation to the ellipse whose focus is (0, 0) directrix is $x + y - 1 = 0$ and $e = \frac{1}{\sqrt{2}}$.

Answer : $3x^2 + 3y^2 - 2xy + 2x + 2y - 1 = 0$.

Standard Equation

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

where $a > b$ & $b^2 = a^2(1 - e^2)$.



Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$

Focii : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

Equations of Directrices : $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

Major Axis : The line segment A'A in which the focii S' & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).

Minor Axis : The y–axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment B'B is of length 2b (b < a) is called the minor axis of the ellipse.

Principal Axis : The major & minor axes together are called principal axis of the ellipse.

Vertices : Point of intersection of ellipse with major axis. $A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

Focal Chord : A chord which passes through a focus is called a focal chord.

Double Ordinate : A chord perpendicular to the major axis is called a double ordinate.

Latus Rectum : The focal chord perpendicular to the major axis is called the latus rectum.

Length of latus rectum **(LL')** = $\frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

= 2e (distance from focus to the corresponding directrix)

Centre : The point which bisects every chord of the conic drawn through it, is called the centre of the conic. $C \equiv (0, 0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Note : (i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned then the rule is to assume that a > b.

(ii) If b > a is given, then the y–axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.

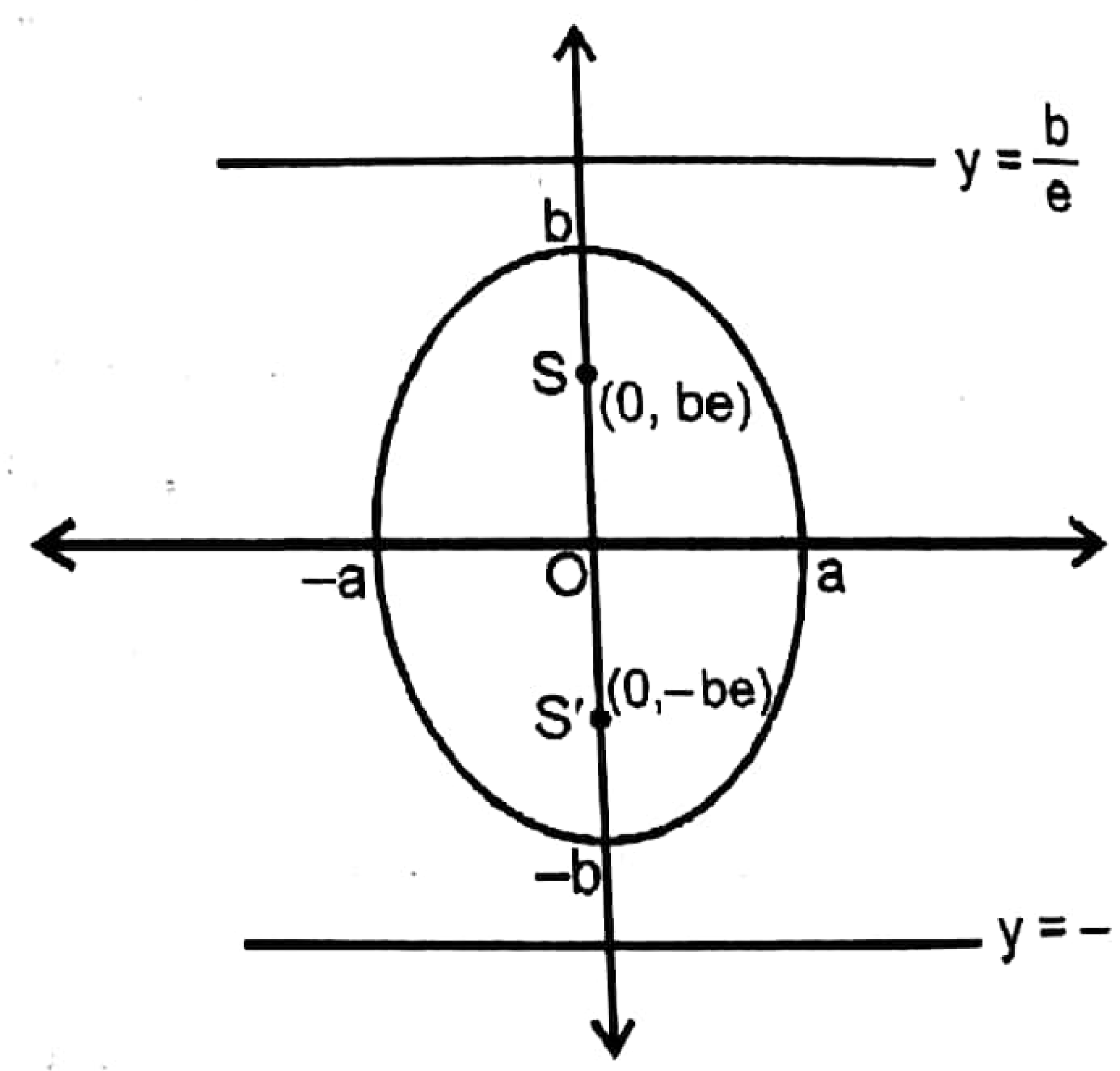
Equation : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Foci $(0, \pm be)$ Directrices : $y = \pm \frac{b}{e}$

$a^2 = b^2(1 - e^2)$, a < b. $\Rightarrow$ $e = \sqrt{1 - \frac{a^2}{b^2}}$

Vertices $(0, \pm b)$; L.R. $y = \pm be$

ℓ (L.R.) = $\frac{2a^2}{b}$, centre : (0, 0)



Example # 2 : Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Solution : Let the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points (2, 2) and (3, 1)

$\therefore$ $\frac{4}{a^2} + \frac{4}{b^2} = 1$ ....(i)

and $\frac{9}{a^2} + \frac{1}{b^2} = 1$ ....(ii) from (i) – 4 (ii), we get

$$\frac{4-36}{a^2} = 1-4 \Rightarrow \qquad a^2 = \frac{32}{3}$$

from (i), we get

$$\frac{1}{b^2} = \frac{1}{4} - \frac{3}{32} = \frac{8-3}{32}$$

$$b^2 = \frac{32}{5}$$

$\therefore$ Ellipse is $3x^2 + 5y^2 = 32$

Example # 3 : Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\frac{1}{3}$

Solution : Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes through centre is minor axis which is y-axis.

Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\because$ $ae = 4$ and $e = \frac{1}{3}$ (Given)

$\therefore$ $a = 12$ and $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = 144\left(1 - \frac{1}{9}\right)$

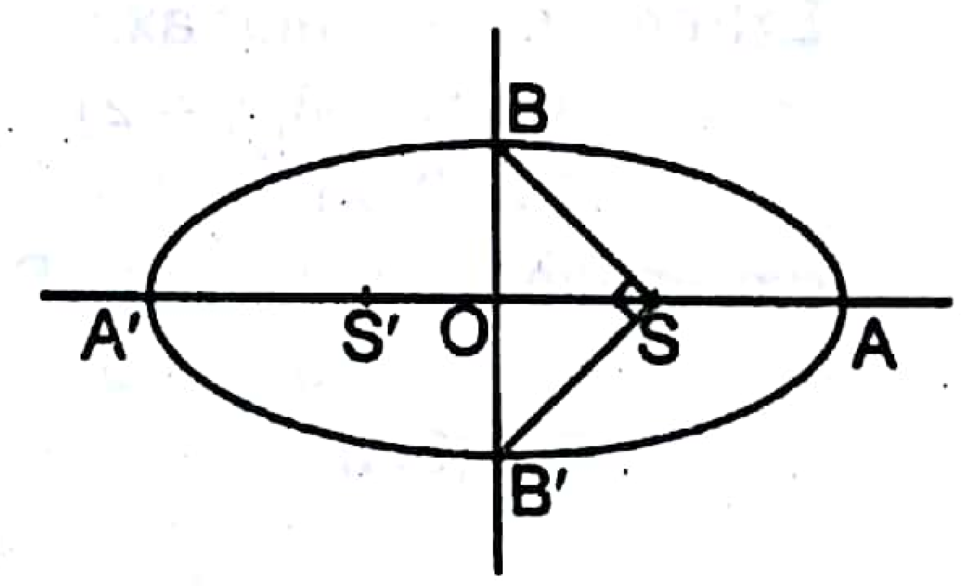
$b^2 = 16 \times 8$

$b = 8\sqrt{2}$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$

Example # 4 : If minor-axis of ellipse subtend a right angle at its focus then find the eccentricity of ellipse.

Solution : Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)



$\because$ $\angle BSB' = \frac{\pi}{2}$

and $OB = OB'$

$\therefore$ $\angle BSO = \frac{\pi}{4}$

$\Rightarrow$ $OS = OB$ $\Rightarrow$ $ae = b$

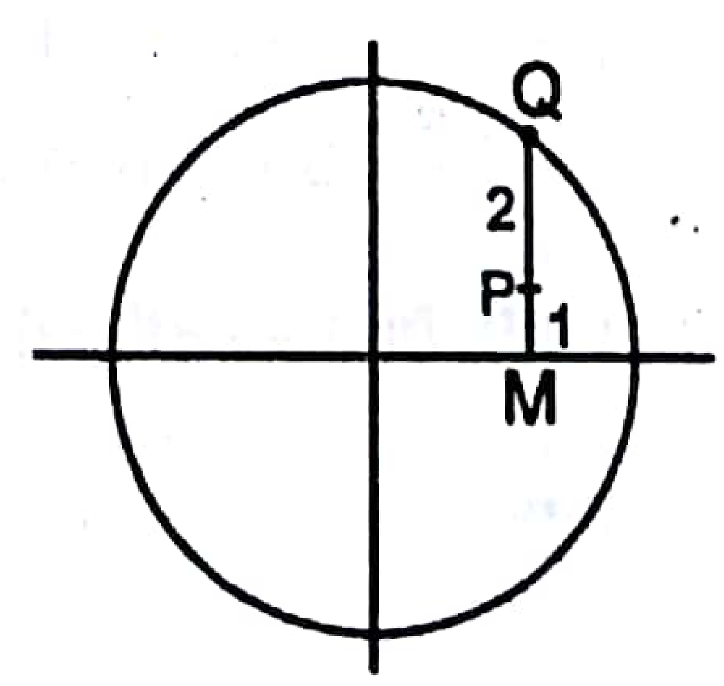
$\Rightarrow$ $e^2 = \frac{b^2}{a^2} = 1 - e^2$ $\Rightarrow$ $e = \frac{1}{\sqrt{2}}$

Example # 5 : From a point Q on the circle $x^2 + y^2 = a^2$, perpendicular QM are drawn to x-axis, find the locus of point 'P' dividing QM in ratio 2 : 1.

Solution : Let $Q \equiv (a\cos\theta, a\sin\theta)$

$M \equiv (a\cos\theta, 0)$

Let $P \equiv (h, k)$



$\therefore$ $h = a\cos\theta,\ k = \frac{a\sin\theta}{3}$

$\therefore \quad \left(\frac{3k}{a}\right)^2 + \left(\frac{h}{a}\right)^2 = 1$

$\Rightarrow \quad$ Locus of P is $\frac{x^2}{a^2} + \frac{y^2}{(a/3)^2} = 1$

Example # 6 : Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus rectum and eccentricity of an ellipse $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$.

Solution : Let $x - 3 = X$, $y - 2 = Y$, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

equation of major axis is $Y = 0 \quad \Rightarrow \quad y = 2$.
equation of minor axis is $X = 0 \quad \Rightarrow \quad x = 3$.
centre $(X = 0, Y = 0) \quad \Rightarrow \quad x = 3, y = 2$
$C \equiv (3, 2)$

Length of semi-major axis $a = 5$
Length of major axis $2a = 10$
Length of semi-minor axis $b = 4$
Length of minor axis $= 2b = 8$.
Let 'e' be eccentricity
$\therefore \quad b^2 = a^2(1 - e^2)$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25 - 16}{25}} = \frac{3}{5}.$$

Length of latus rectum $= LL' = \frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$

Co-ordinates focii are $X = \pm ae$, $Y = 0$
$\Rightarrow \quad S \equiv (X = 3, Y = 0) \quad \& \quad S' \equiv (X = -3, Y = 0)$
$\Rightarrow \quad S \equiv (6, 2) \quad \& \quad S' \equiv (0, 2)$

<u>**Co-ordinate of vertices**</u>

Extremities of major axis $A \equiv (X = a, Y = 0) \quad \& \quad A' \equiv (X = -a, Y = 0)$
$\Rightarrow \quad A \equiv (x = 8, y = 2) \quad \& \quad A' = (x = -2, 2)$
$A \equiv (8, 2) \quad \& \quad A' \equiv (-2, 2)$
Extremities of minor axis $B \equiv (X = 0, Y = b) \quad \& \quad B' \equiv (X = 0, Y = -b)$
$B \equiv (x = 3, y = 6) \quad \& \quad B' \equiv (x = 3, y = -2)$
$B \equiv (3, 6) \quad \& \quad B' \equiv (3, -2)$

Equation of directrix $X = \pm \frac{a}{e}$

$x - 3 = \pm \frac{25}{3} \quad \Rightarrow \quad x = \frac{34}{3} \quad \& \quad x = -\frac{16}{3}$

Self Practice Problem

(2) Find the equation to the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ and their equations are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$ respectively.

(3) Find the eccentricity of ellipse whose minor axis is double the latus rectum.

(4) Find the co-ordinates of the focii of the ellipse $4x^2 + 9y^2 = 1$.

(5) Find the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passing through (2, 1) and having eccentricity $\frac{1}{2}$.

(6) A point moves so that the sum of the squares of its distances from two intersecting lines is constant (given that the lines are neither perpendicular nor they make complimentry angle). Prove that its locus is an ellipse.

Hint. : Assume the lines to be $y = mx$ and $y = -mx$.

Answers : (2) $3(x - 3y + 3)^2 + 2(3x + y - 1)^2 = 180$,

$21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$.

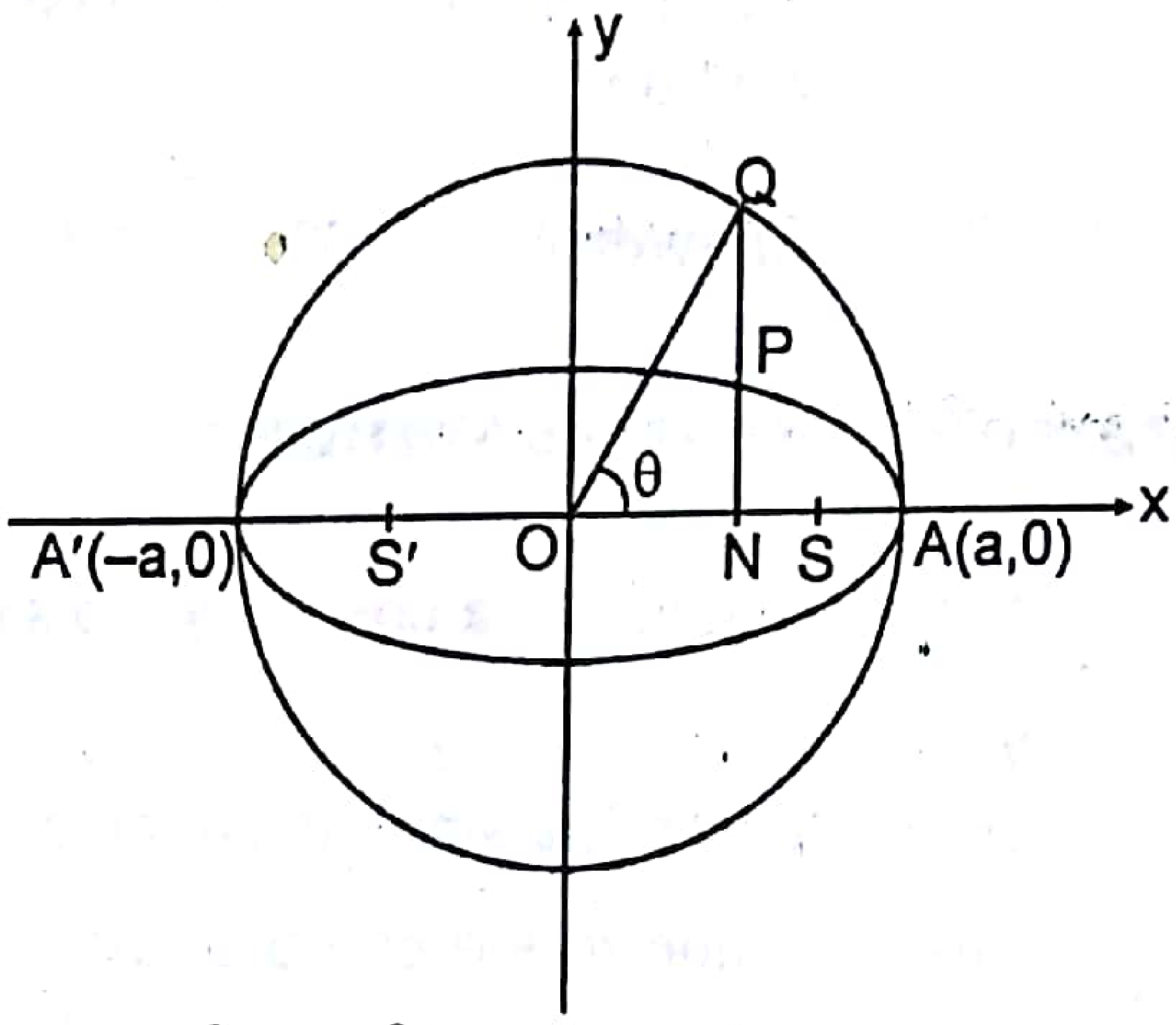
(3) $\frac{\sqrt{3}}{2}$ (4) $\left(\pm\frac{\sqrt{5}}{6}, 0\right)$ (5) $3x^2 + 4y^2 = 16$

Auxiliary Circle / Eccentric Angle :

A circle described on major axis of ellipse as diameter is called the **auxiliary circle.**

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x – axis on the way intersects the ellipse at P, then P & Q are called as the **Corresponding Points** on the ellipse & the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse $(-\pi < \theta \le \pi)$. $Q \equiv (a\cos\theta, a\sin\theta)$

$P \equiv (a\cos\theta, b\sin\theta)$



Note that :

$$\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$$

NOTE : If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

Example # 7 : Find the focal distance of a point P(θ) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)

Solution : Let 'e' be the eccentricity of ellipse.

$\therefore$ PS = e . PM

$= e\left(\frac{a}{e} - a\cos\theta\right)$

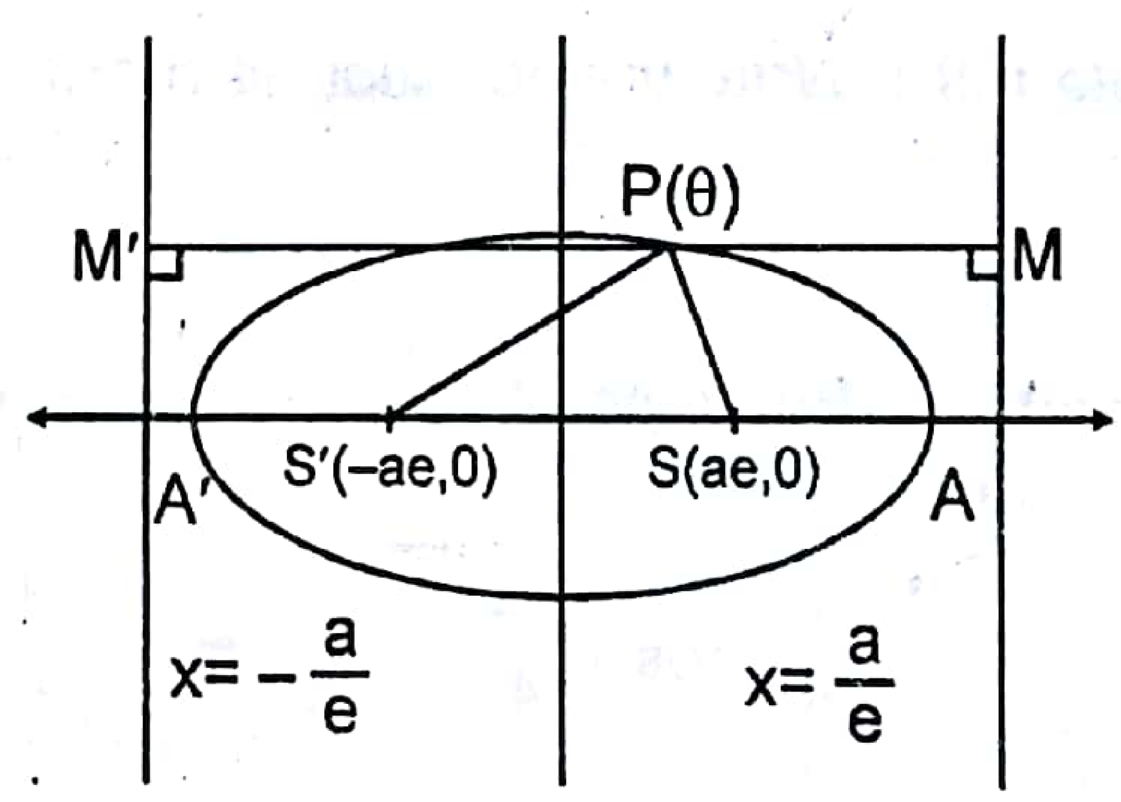
PS = (a – a e cosθ)

and PS′ = e. PM′

$= e\left(a\cos\theta + \frac{a}{e}\right)$

PS′ = a + ae cosθ

$\therefore$ focal distance are $(a \pm ae\cos\theta)$



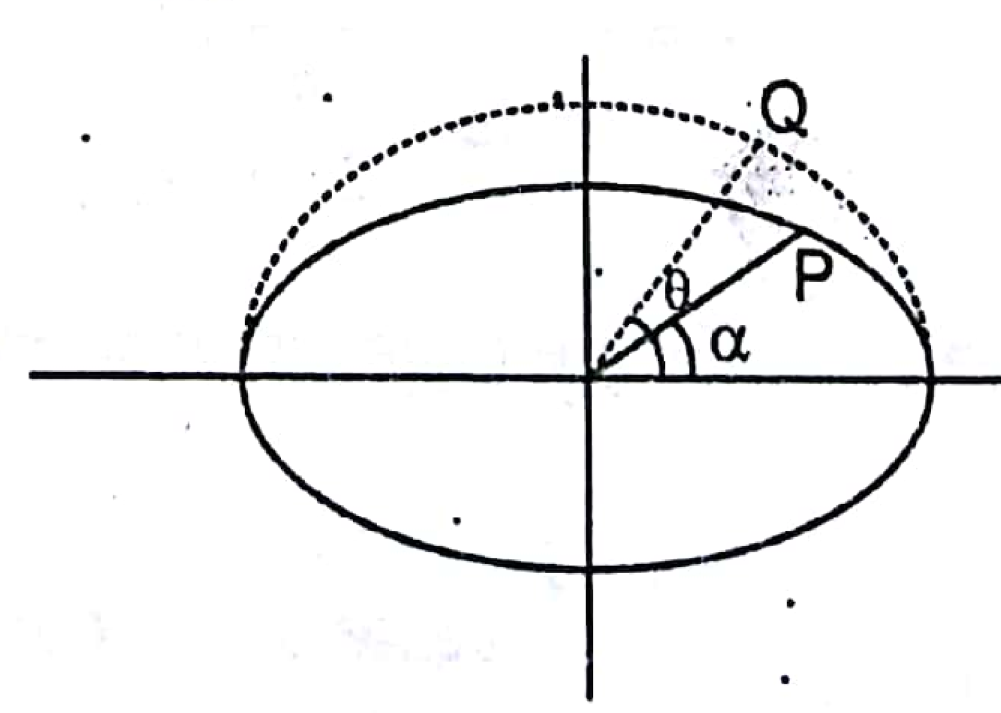
Note : PS + PS′ = 2a

PS + PS′ = AA′

Eample # 8 : Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose radius makes angle α with x-axis.

Solution : Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore m_{(op)} = \frac{b}{a}\tan\theta = \tan\alpha \Rightarrow \tan\theta = \frac{a}{b}\tan\alpha$



$$OP = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \sqrt{\frac{a^2 + b^2\tan^2\theta}{\sec^2\theta}}$$

$$= \sqrt{\frac{a^2+b^2\tan^2\theta}{1+\tan^2\theta}} = \sqrt{\frac{a^2+b^2\times\frac{a^2}{b^2}\tan^2\alpha}{1+\frac{a^2}{b^2}\tan^2\alpha}}$$

$$OP = \frac{ab}{\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}}$$

Self Practice Problem

(7) Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ whose eccentric angle is α

(8) Find the eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2} = 1$ whose distance from the centre is 2.

(9) Show that the area of triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle.

Answers : (7) $r = \sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$ (8) $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

Parametric Representation :

The equations x = a cos θ & y = b sin θ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if P(θ) ≡ (a cos θ, b sin θ) is on the ellipse then;
Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

Example # 9 : Write the equation of chord of an ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

Solution : Equation of chord is $\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}+\frac{y}{4}.\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} = \cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$

$$\frac{x}{5}.\cos\left(\frac{3\pi}{4}\right)+\frac{y}{4}.\sin\left(\frac{3\pi}{4}\right) = 0$$

$$-\frac{x}{5}+\frac{y}{4} = 0 \quad\Rightarrow\quad 4x = 5y$$

Example # 10 : If P(α) and P(β) are extremities of a focal chord of ellipse then prove that its eccentricity

$$e = \left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|.$$

Solution : Let the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

∴ equation of chord is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

Since above chord is focal chord,

$\therefore$ it passes through focus (ae, 0) or (– ae, 0)

$$\therefore \quad \pm e \cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

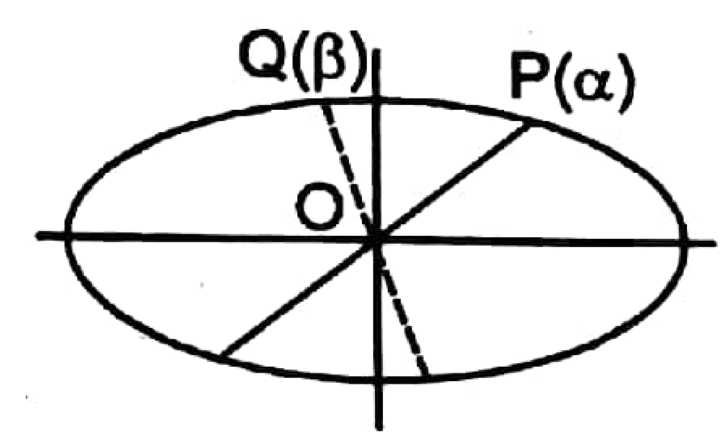
$$\therefore \quad e = \left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right| \qquad \textbf{Ans.}$$

Note : $\because \quad \pm e = \dfrac{\cos\dfrac{\alpha-\beta}{2}}{\cos\dfrac{\alpha+\beta}{2}} \quad \Rightarrow \quad \pm e = \dfrac{1+\tan\dfrac{\alpha}{2}.\tan\dfrac{\beta}{2}}{1-\tan\dfrac{\alpha}{2}.\tan\dfrac{\beta}{2}}$

Applying componendo and dividendo $\dfrac{1\pm e}{\pm e-1} = \dfrac{2}{2\tan\dfrac{\alpha}{2}.\tan\dfrac{\beta}{2}}$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1+e}{e-1} \text{ or } \frac{e-1}{1+e}$$

Example # 11 : Find the angle between two diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$. Whose extremities have eccentricity angle α and $\beta = \alpha + \dfrac{\pi}{2}$.



Solution : Let ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$

Slope of OP = $m_1 = \dfrac{b\sin\alpha}{a\cos\alpha} = \dfrac{b}{a}\tan\alpha$

Slope of OQ = $m_2 = \dfrac{b\sin\beta}{a\cos\beta} = -\dfrac{b}{a}\cot\alpha$ given $\beta = \alpha + \dfrac{\pi}{2}$

$$\therefore \quad \tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{\frac{b}{a}(\tan\alpha+\cot\alpha)}{1-\frac{b^2}{a^2}}\right| = \left|\frac{2ab}{(a^2-b^2)\sin 2\alpha}\right|$$

Self Practice Problem :

(10) Find the sum of squares of two diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ whose extremities have eccentric angles differ by $\dfrac{\pi}{2}$ and show that it is constant.

(11) Show that the sum of squares of reciprocals of two perpendicular diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ is constant. Find the constant also.

(12) Find the locus of the foot of the perpendicular from the centre of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ on the chord joining two points whose eccentric angles differ by $\dfrac{\pi}{2}$.

Answers : (10) $4(a^2 + b^2)$ (11) $\dfrac{1}{4}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$ (12) $2(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2$.

Position of a Point w.r.t. an Ellipse :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ > < or = 0.

Example#12 : Check whether the point P(3, 2) lies inside or outside of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Solution : $S_1 \equiv \frac{9}{25}+\frac{4}{16}-1=\frac{9}{25}+\frac{1}{4}-1<0$

$\therefore$ Point $P \equiv (3, 2)$ lies inside the ellipse.

Example#13: Find the set of value(s) of 'α' for which the point $P(\alpha, -\alpha)$ lies inside the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Solution : If $P(\alpha, -\alpha)$ lies inside the ellipse

$\therefore$ $S_1 < 0$

$\Rightarrow \frac{\alpha^2}{16}+\frac{\alpha^2}{9}-1<0 \Rightarrow \frac{25}{144}\cdot\alpha^2<1 \Rightarrow \alpha^2<\frac{144}{25}$

$\therefore$ $\alpha \in \left(-\frac{12}{5}, \frac{12}{5}\right)$.

Line and an Ellipse :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.

Example#14 : Find the set of value(s) of 'λ' for which the line $3x - 4y + \lambda = 0$ intersect the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at two distinct points.

Solution : Solving given line with ellipse, we get $\frac{(4y-\lambda)^2}{9\times16}+\frac{y^2}{9}=1$

$\frac{2y^2}{9}-\frac{y\lambda}{18}+\frac{\lambda^2}{144}-1=0$

Since, line intersect the parabola at two distinct points,

$\therefore$ roots of above equation are real & distinct

$\therefore$ $D > 0$

$\Rightarrow \frac{\lambda^2}{(18)^2}-\frac{8}{9}\cdot\left(\frac{\lambda^2}{144}-1\right)>0 \Rightarrow -12\sqrt{2}<\lambda<12\sqrt{2}$

Self Practice Problem

(13) Find the value of 'λ' for which $2x - y + \lambda = 0$ touches the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$

Answer: $\lambda = \pm\sqrt{109}$

Tangents :

(a) Slope form: $y = mx \pm \sqrt{a^2m^2+b^2}$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m.

(b) Point form : $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1).

(c) Parametric form: $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$.

Note : (i) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction. These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents at the point α & β is, $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(iii) The eccentric angles of the points of contact of two parallel tangents differ by π.

Example # 15 : Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Solution : Slope of tangent = m = $\frac{1}{2}$

Given ellipse is $\frac{x^2}{4}+\frac{y^2}{3}=1$

Equation of tangent whose slope is 'm' is $y = mx \pm \sqrt{4m^2+3}$

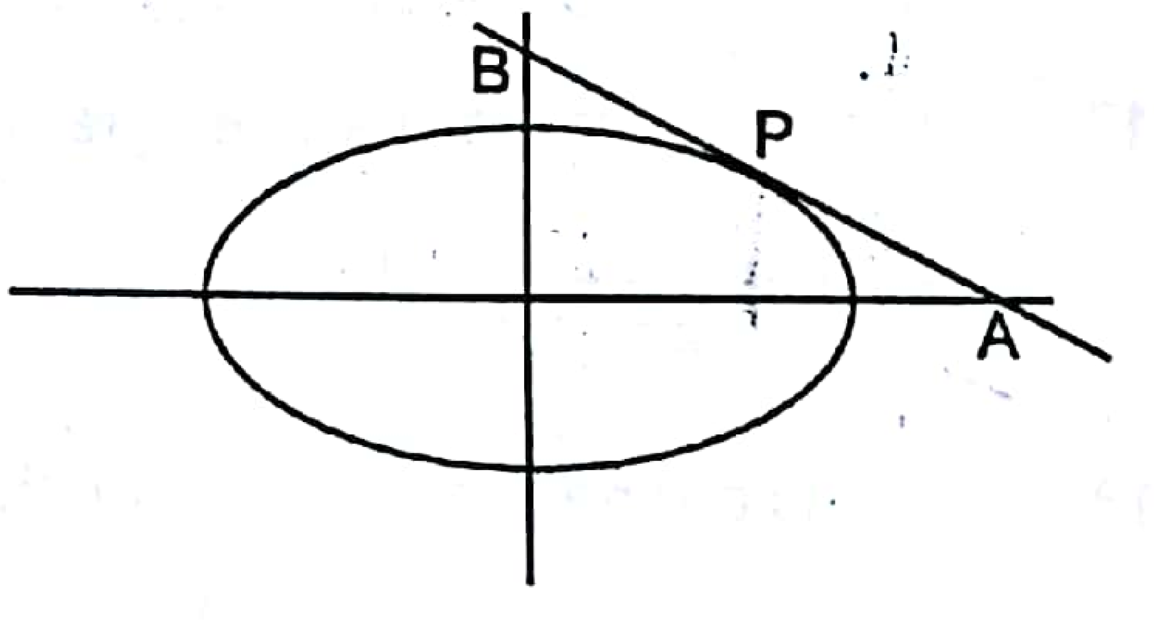
$\because$ $m = \frac{1}{2}$ $\quad\therefore\quad$ $y = \frac{1}{2}x \pm \sqrt{1+3}$

$2y = x \pm 4$

Example # 16 : A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Solution : Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore$ equation of tangent is



$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$A \equiv (a\sec\theta, 0)$

$B \equiv (0, b\,\mathrm{cosec}\theta)$

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore$ $a\cos\theta = \frac{a\sec\theta}{4}$ $\Rightarrow$ $\cos^2\theta = \frac{1}{4}$ $\Rightarrow$ $\cos\theta = \frac{1}{2}$

and $b\sin\theta = \frac{3b\,\mathrm{cosec}\theta}{4}$ $\Rightarrow$ $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ tangent is $\frac{x}{2a}+\frac{\sqrt{3}y}{2b}=1$ $\Rightarrow$ $bx + \sqrt{3}\,ay = 2ab$

Example # 17 : Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angle differ by a constant α is an ellipse.

Solution : Let P (h, k) be the point of intersection of tangents at $A(\theta)$ and $B(\beta)$ to the ellipse.

$$\therefore\quad h = \frac{a\cos\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \;\&\; k = \frac{b\sin\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \Rightarrow \left(\frac{h}{a}\right)^2+\left(\frac{k}{b}\right)^2 = \sec^2\left(\frac{\theta-\beta}{2}\right)$$

but given that $\theta - \beta = \alpha$

$$\therefore\quad \text{locus is } \frac{x^2}{a^2\sec^2\left(\frac{\alpha}{2}\right)}+\frac{y^2}{b^2\sec^2\left(\frac{\alpha}{2}\right)}=1$$

Example#18 : Find the locus of foot of perpendicular drawn from centre to any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : Let P(h, k) be the foot of perpendicular to a tangent $y = mx + \sqrt{a^2m^2 + b^2}$ ....(i)
from centre

$\therefore \quad \frac{k}{h} \cdot m = -1 \quad \Rightarrow \quad m = -\frac{h}{k}$ ....(ii)

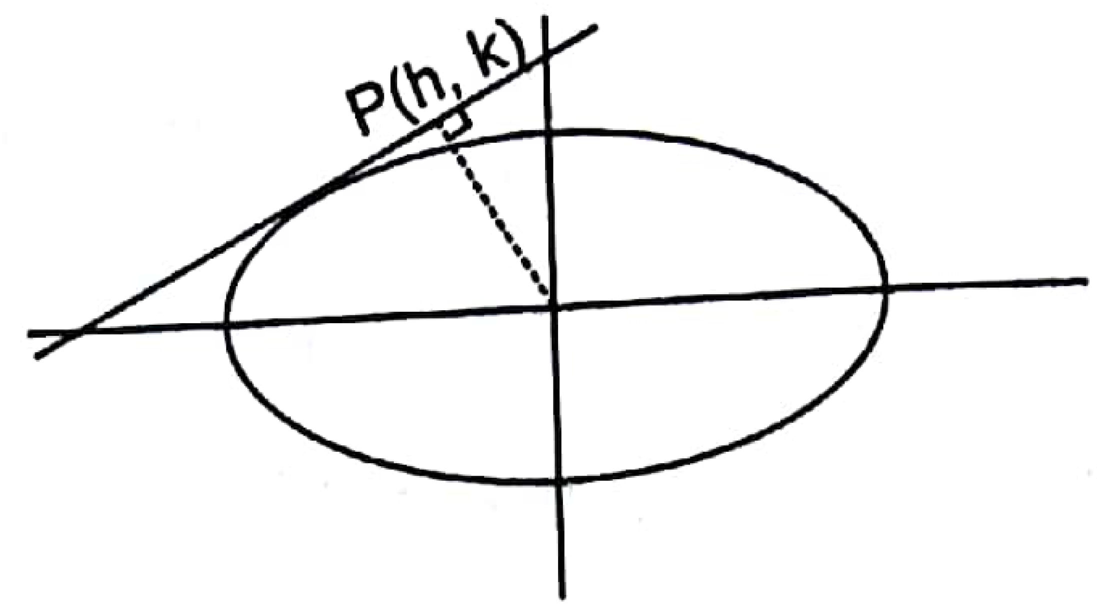
$\because$ P(h, k) lies on tangent

$\therefore \quad k = mh + \sqrt{a^2m^2 + b^2}$ ....(iii)

from equation (ii) & (iii), we get

$$\left(k + \frac{h^2}{k}\right)^2 = \frac{a^2h^2}{k^2} + b^2$$

$\Rightarrow$ locus is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$



Self Practice Problem

(14) Show that the locus of the point of intersection of the tangents at the extremities of any focal chord of an ellipse is the directrix corresponding to the focus.

(15) Show that the locus of the foot of the perpendicular on a varying tangent to an ellipse from either of its foci is a concentric circle.

(16) Prove that the portion of the tangent to an ellipse intercepted between the ellipse and the directrix subtends a right angle at the corresponding focus.

(17) Find the area of parallelogram formed by tangents at the extremities of latera recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(18) If y_1 is ordinate of a point P on the ellipse then show that the angle between its focal radius and tangent at it, is $\tan^{-1}\left(\frac{b^2}{aey_1}\right)$.

(19) Find the eccentric angle of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ tangent at which, is equally inclined to the axes.

Answers : (17) $\frac{2a^3}{\sqrt{a^2 - b^2}}$ (19) $\theta = \pm \tan^{-1}\left(\frac{b}{a}\right), \pi - \tan^{-1}\left(\frac{b}{a}\right), -\pi + \tan^{-1}\left(\frac{b}{a}\right)$

Normals :

(i) Equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

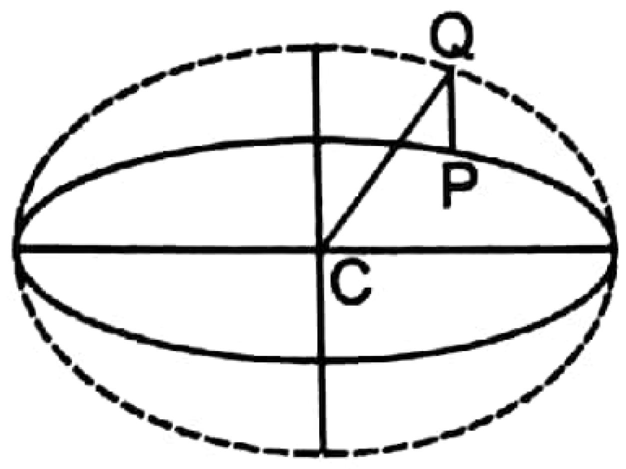
(ii) Equation of the normal at the point (acos θ· bsin θ) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is;
$ax.\sec\theta - by\cdot\mathrm{cosec}\,\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

Example # 19 : P and Q are corresponding points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

Solution : Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore$ $Q \equiv (a\cos\theta, a\sin\theta)$



Equation of normal at P is

$(a\sec\theta)x - (b\,\text{cosec}\,\theta)y = a^2 - b^2$ ....(i)

equation of CQ is $y = \tan\theta \cdot x$ ....(ii)

Solving equation (i) & (ii), we get $(a - b)x = (a^2 - b^2)\cos\theta$

$x = (a + b)\cos\theta$, & $y = (a + b)\sin\theta$

$\therefore$ $R \equiv ((a + b)\cos\theta, (a + b)\sin\theta)$

$\therefore$ $CR = a + b$

Example # 20 : Find the shortest distance between the line x + y = 10 and the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Solution : Shortest distance occurs between two non-intersecting curve always along common normal.

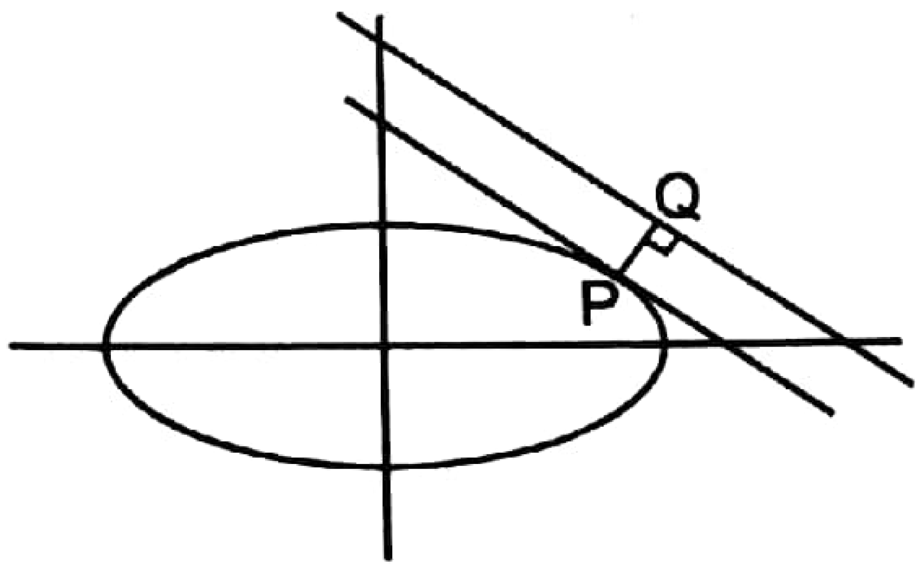
Let 'P' be a point on ellipse and Q is a point on given line for which PQ is common normal.

$\therefore$ Tangent at 'P' is parallel to given line

$\therefore$ Equation of tangent parallel to given line is $(y = mx \pm \sqrt{a^2m^2 + b^2})$

$y = -x \pm 5$

$\Rightarrow$ $x + y + 5 = 0$ or $x + y - 5 = 0$



$\therefore$ minimum distance = distance between

$x + y - 10 = 0$ & $x + y - 5 = 0$

$\Rightarrow$ shortest distance $= \frac{|10 - 5|}{\sqrt{1+1}} = \frac{5}{\sqrt{2}}$

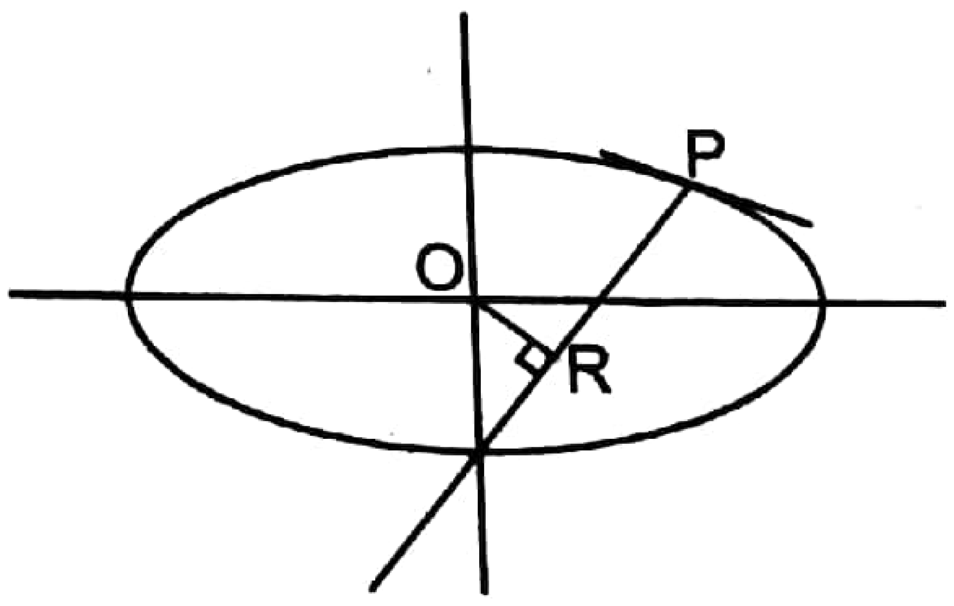
Example # 21 : Prove that, in an ellipse, the distance between the centre and any normal does not exceed the difference between the semi-axes of the ellipse.

Solution : Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of normal at P (θ) is $(a\sec\theta)x - (b\,\text{cosec}\,\theta)y - a^2 + b^2 = 0$

distance of normal from centre

$$= OR = \frac{|a^2 - b^2|}{\sqrt{a^2 + b^2 + (a\tan\theta)^2 + (b\cot\theta)^2}}$$



$$= \frac{|a^2 - b^2|}{\sqrt{(a+b)^2 + (a\tan\theta - b\cot\theta)^2}}$$

$\because$ $(a + b)^2 + (a\tan\theta - b\cot\theta)^2 \geq (a + b)^2$ or $\leq \frac{|a^2 - b^2|}{\sqrt{(a+b)^2}}$

$|OR| \leq (a - b)$

Self Practice Problem

(20) Find the value(s) of 'k' for which the line x + y = k is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(21) If the normal at the point P(θ) to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersects it again at the point Q(2θ) then cosθ =

(A) $-\frac{2}{3}$ (B) $\frac{2}{3}$ (C) $-\frac{6}{7}$ (D) $\frac{6}{7}$

Answers : (20) $k = \pm\sqrt{\frac{(a^2 - b^2)^2}{a^2 + b^2}}$ (21) A

Pair of Tangents :

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is given by: $SS_1 = T^2$ where :

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2} - 1 \quad ; \quad S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} - 1 \; ; \quad T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1.$$

Example # 22 : How many real tangents can be drawn from the point (4, 3) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

Solution : Given point $P \equiv (4, 3)$

ellipse $S \equiv \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$

$\because \quad S_1 \equiv \frac{16}{16} + \frac{9}{9} - 1 = 1 > 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the ellipse.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is $SS_1 = T^2$

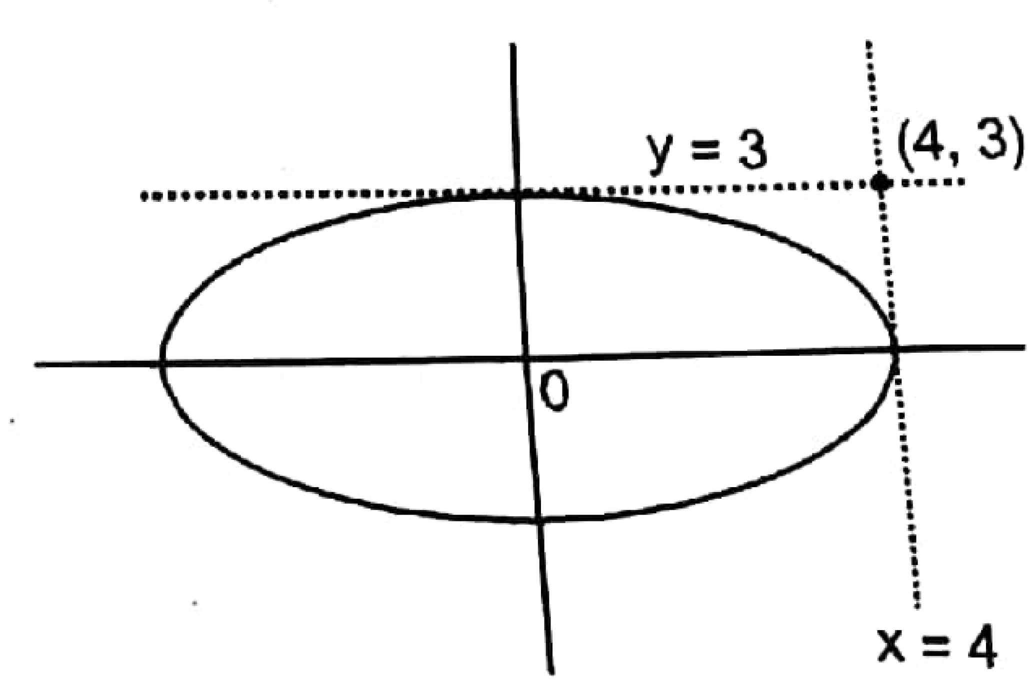
$$\Rightarrow \left(\frac{x^2}{16}+\frac{y^2}{9}-1\right) . 1 = \left(\frac{4x}{16}+\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} - 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 + \frac{xy}{6} - \frac{x}{2} - \frac{2y}{3}$$

$\Rightarrow \quad - xy + 3x + 4y - 12 = 0 \quad \Rightarrow \quad (4 - x)(y - 3) = 0$

$\Rightarrow \quad x = 4$ & $y = 3$

and angle between them = $\frac{\pi}{2}$



Alternative

By direct observation

$x = 4, \; y = 3$ are tangents.

Example # 23 : Find the locus of point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right) = \left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right) + \ldots\ldots = 0 \qquad \ldots\ldots\ldots(i)$$

Since equation (i) represents two perpendicular lines

$$\therefore \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right) + \frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right) = 0$$

$\Rightarrow \quad k^2 - b^2 + h^2 - a^2 = 0 \quad \Rightarrow \quad$ locus is $x^2 + y^2 = a^2 + b^2$

(22) Find the locus of point of intersection of the tangents drawn at the extremities of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer : $x = \pm \frac{a}{e}$

Director Circle :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axes.

Example # 24 : An ellipse slides between two perpendicular lines. Show that the locus of its centre is a circle.

Solution : Let length of semi-major and semi-minor axis are 'a' and 'b' and centre is $C \equiv (h, k)$

Since ellipse slides between two perpendicular lines, therefore point of intersection of two perpendicular tangents lies on director circle.

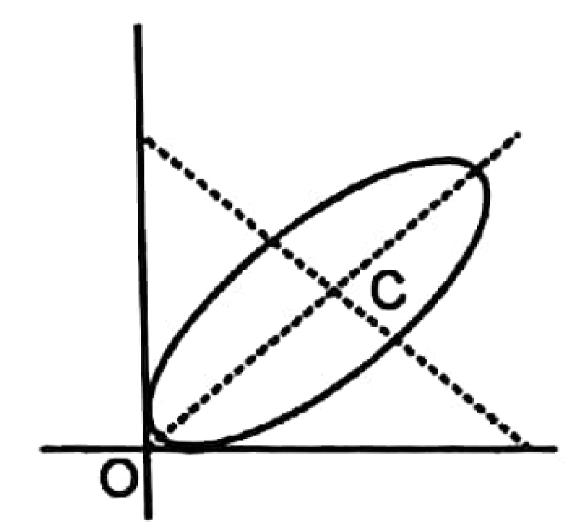
Let us consider two perpendicular lines as x & y axes

$\therefore$ point of intersection is origin $O \equiv (0, 0)$

$\therefore$ OC = radius of director circle

$\therefore$ $\sqrt{h^2 + k^2} = \sqrt{a^2 + b^2}$

$\Rightarrow$ locus of $C \equiv (h, k)$ is $\Rightarrow$ $x^2 + y^2 = a^2 + b^2$ which is a circle



Self Practice Problem

(23) A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Chord of Contact :

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

T = 0, where $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

Example # 25 : If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution : Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

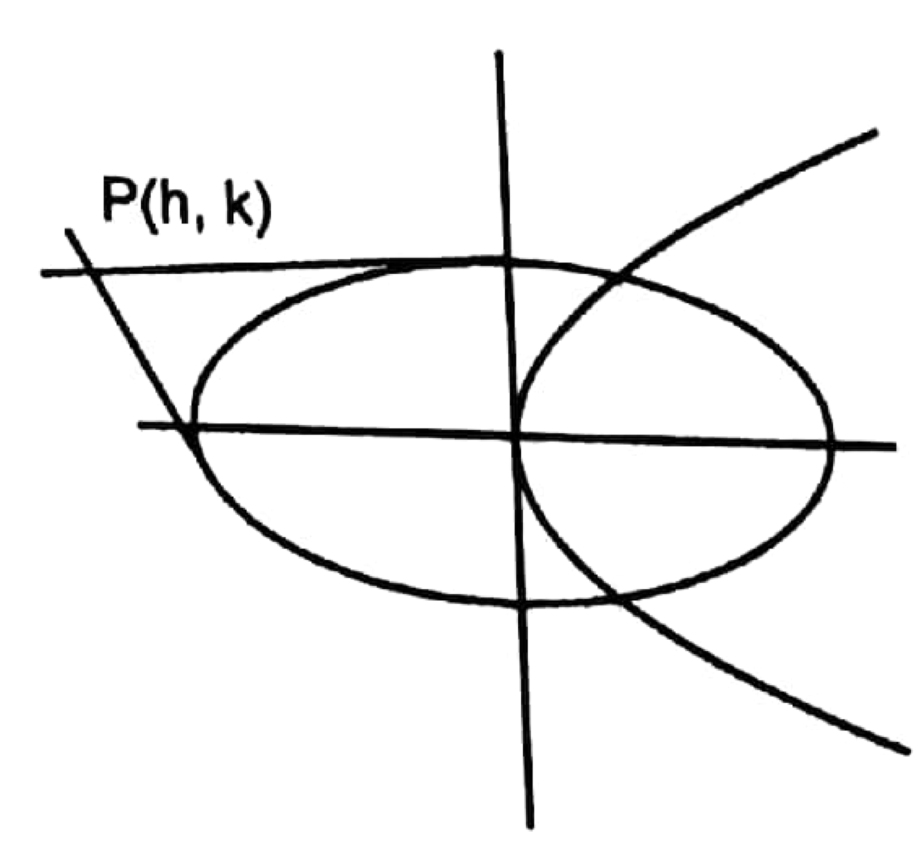
$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ ....(i)

which touches the parabola equation of tangent to parabola $y^2 = 4ax$

$$y = mx + \frac{a}{m}$$

$\Rightarrow$ $mx - y = -\frac{a}{m}$ ....(ii)

equation (i) & (ii) as must be same



$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1}$$

$$\Rightarrow \quad m = -\frac{h}{k}\frac{b^2}{a^2} \ \& \ m = \frac{ak}{b^2}$$

$$\therefore \quad -\frac{hb^2}{ka^2} = \frac{ak}{b^2} \quad \Rightarrow \quad \text{locus of P is } y^2 = -\frac{b^4}{a^3} \cdot x$$

Self Practice Problem

(24) Find the locus of point of intersection of tangents at the extremities of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(25) Find the locus of point of intersection of tangents at the extremities of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtending a right angle at its centre.

Answers : (24) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$ (25) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$

Chord with a given middle point :

Equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose middle point is (x_1, y_1) is $T = S_1$,

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$; $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$.

Example # 26 : Find the locus of the mid - point of focal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : Let $P \equiv (h, k)$ be the mid-point

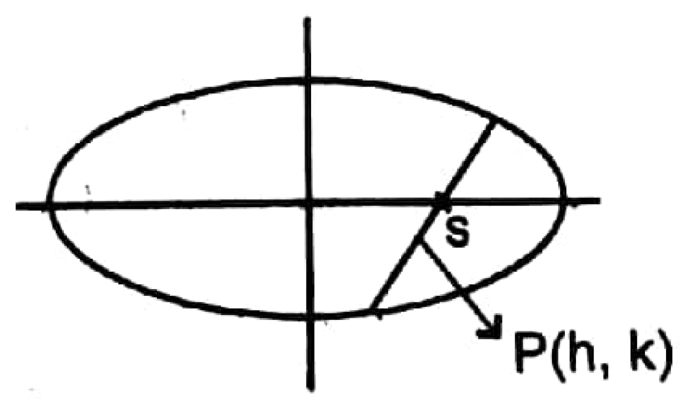
$\therefore$ equation of chord whose mid-point is given $\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ it passes through focus, either (ae, 0) or (–ae, 0)

If it passes through (ae, 0)

$\therefore$ locus is $\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



If it passes through (–ae, 0)

$\therefore$ locus is $-\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Example # 27 : Find the condition on 'a' and 'b' for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through (a, – b) are bisected by the line x + y = b.

Solution : Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ equation of chord whose mid-point is $P(\alpha, b - \alpha)$

$$\frac{x\alpha}{2a^2} + \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} + \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, –b)

$$\therefore \quad \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} + \frac{(b-\alpha)^2}{2b^2}$$

$$\Rightarrow \quad \left(\frac{1}{a}+\frac{1}{b}\right)\alpha - 1 = \alpha^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - \frac{2}{b}\alpha + 1$$

$\Rightarrow \quad \alpha^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\left(\frac{3}{b}+\frac{1}{a}\right)\alpha+2=0$

since line bisect two chord

$\therefore$ above quadratic equation in α must have two distinct real roots

$\therefore \quad \left(\frac{3}{b}+\frac{1}{a}\right)^2-4\left(\frac{1}{a^2}+\frac{1}{b^2}\right).2>0$

$\Rightarrow \quad \frac{9}{b^2}+\frac{1}{a^2}+\frac{6}{ab}-\frac{8}{a^2}-\frac{8}{b^2}>0 \quad \Rightarrow \quad \frac{1}{b^2}-\frac{7}{a^2}+\frac{6}{ab}>0$

$\Rightarrow \quad a^2-7b^2+6ab>0$

$\Rightarrow \quad a^2>7b^2-6ab$ which is the required condition.

Self Practice Problem

(26) Find the equation of the chord $\frac{x^2}{36}+\frac{y^2}{9}=1$ which is bisected at (2, 1).

(27) Find the locus of the mid-points of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(28) Find the length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ whose middle point is $\left(\frac{1}{2},\frac{2}{5}\right)$

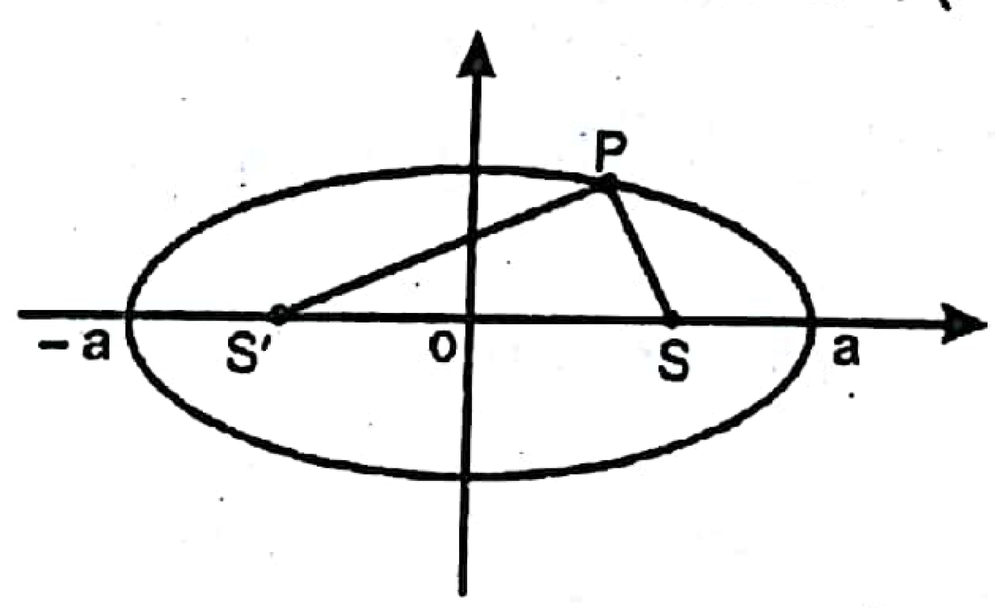
Answers : (26) $x+2y=4$ (27) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2$

(28) $\frac{7}{5}\sqrt{41}$

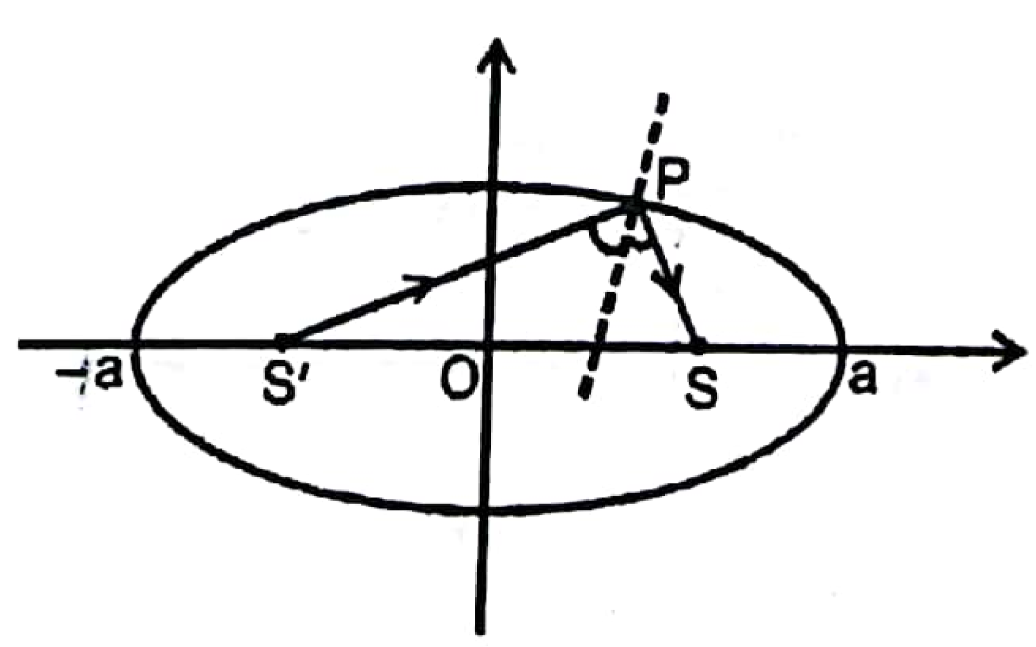
Important Highlights :

Refering to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

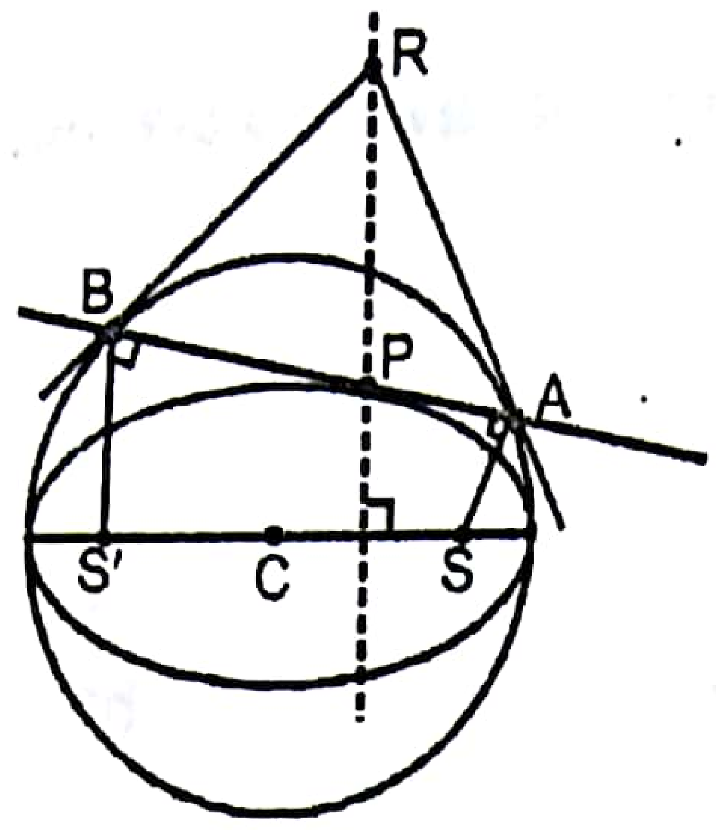
(1) If P be any point on the ellipse with S & S′ as its foci then ℓ(SP) + ℓ(S′P) = 2a.



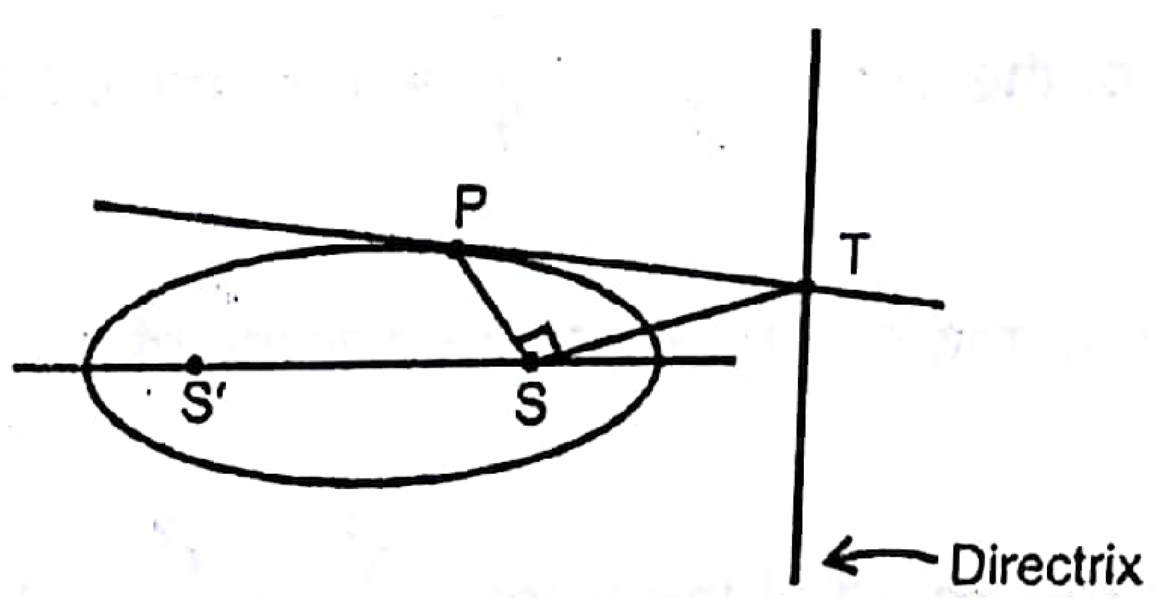
(2) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.



(3) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle.



(4) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

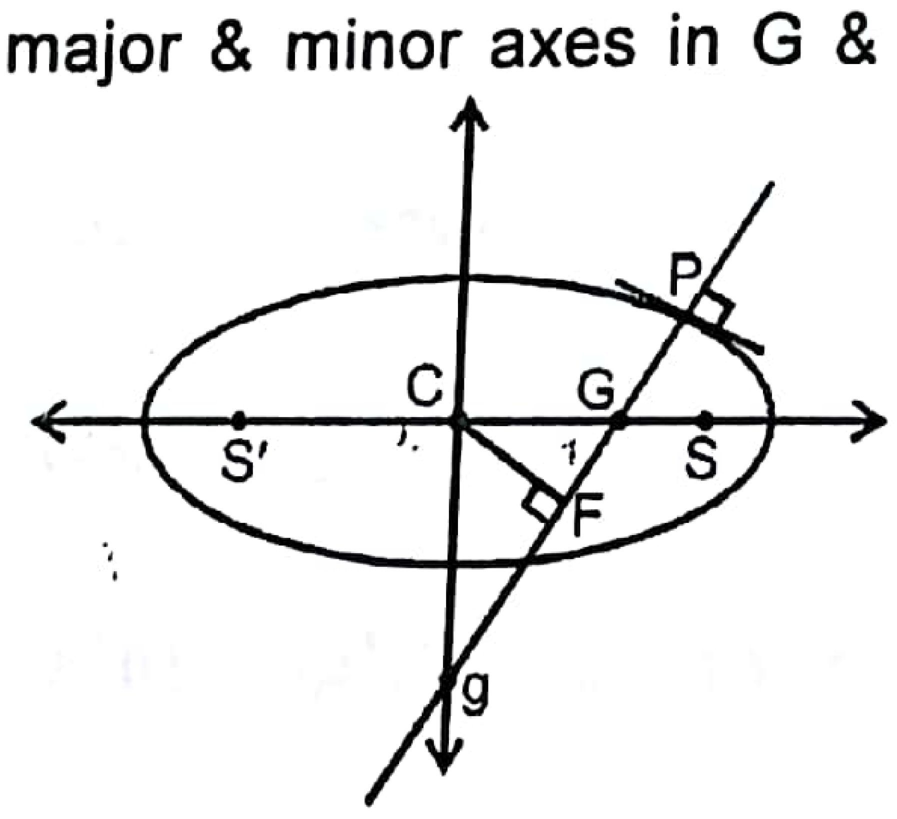


(5) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively & if CF be perpendicular upon this normal then

(i) PF. PG = b^2

(ii) PF. Pg = a^2

(iii) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.



[where S and S′ are the focii of the ellipse]

(6) The circle on any focal distance as diameter touches the auxiliary circle.

Exercise # 1

Single choice type

Section (A) : Ellipse and Terminology, Focal Property

A-1. The equation of the ellipse whose focus is (1, –1), directrix is the line $x - y - 3 = 0$ and the eccentricity is $\frac{1}{2}$, is

(1) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$ (2) $7x^2 + 2xy + 7y^2 + 7 = 0$
(3) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$ (4) none of these

A-2. The eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is

(1) $\frac{5}{6}$ (2) $\frac{3}{5}$ (3) $\frac{\sqrt{2}}{3}$ (4) $\frac{\sqrt{5}}{3}$

A-3. If distance between the directrices be thrice the distance between the focii, then eccentricity of ellipse is

(1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) $\frac{1}{\sqrt{3}}$ (4) $\frac{4}{5}$

A-4. The length of the latus rectum of the ellipse $9x^2 + 4y^2 = 1$, is

(1) $\frac{3}{2}$ (2) $\frac{8}{3}$ (3) $\frac{4}{9}$ (4) $\frac{8}{9}$

A-5. The eccentricity of the ellipse which meets the straight line $\frac{x}{7} + \frac{y}{2} = 1$ on the axis of x and the straight line $\frac{x}{3} - \frac{y}{5} = 1$ on the axis of y and whose axes lie along the axes of coordinates is

(1) $\frac{\sqrt{6}}{7}$ (2) $\frac{4\sqrt{6}}{7}$ (3) $\frac{2\sqrt{6}}{5}$ (4) $\frac{2\sqrt{6}}{7}$

A-6. Let P be a variable point on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ with focii at S and S′. If A be the area of triangle PSS′, then the maximum value of A is

(1) 24 sq. units (2) 12 sq. units (3) 36 sq. units (4) none of these

A-7. The ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ have in common

(1) centre only (2) centre, foci and directries
(3) Centre, foci and vertices (4) centre and vertices only

A-8. Equation of the ellipse whose foci are (2, 2) and (4, 2) and the major axis is of length 10 is

(1) $\frac{(x+3)^2}{4} + \frac{(y+2)^2}{5} = 1$ (2) $\frac{(x+3)^2}{24} + \frac{(y+2)^2}{25} = 1$

(3) $\frac{(x+3)^2}{25} + \frac{(y+2)^2}{24} = 1$ (4) $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{24} = 1$

A-9. The length of the axes of the conic $9x^2 + 4y^2 - 6x + 4y + 1 = 0$, are

(1) $\frac{1}{2}$, 9 (2) 3, $\frac{2}{5}$ (3) 1, $\frac{2}{3}$ (4) 3, 2

A-10. The equation $\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0$ represents an ellipse, if

(1) $r > 2$ (2) $2 < r < 5$ (3) $r > 5$ (4) $r \in \{2, 5\}$

A-11. An arc of a bridge is semi-elliptical with major axis horizontal. The length of the base is 9 meter and the highest part of the bridge is 3 meter from the horizontal. The best approximation of the length of the Pillar 2 meter from the centre of the base is :

(1) 11/4 m (2) 8/3 m (3) 7/2 m (4) 2 m

A-12. An ellipse has OB as semi-minor axis, F and F′ are its foci and $\angle FBF'$ is a right angle then eccentricity of the ellipse is

(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{2}{3}$ (4) $\frac{1}{3}$

A-13. The eccentricity of an ellipse in which distance between their focii is 10 and that of focus and corresponding directrix is 15 is

(1) $\frac{1}{3}$ (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{2}}$

A-14. If $P = (x, y)$, $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals

(1) 8 (2) 6 (3) 10 (4) 12

.. If focus and corresponding directrix of an ellipse are (3, 4) and $x + y - 1 = 0$ and eccentricity is $\frac{1}{2}$ then the co-ordinates of extremities of major axis are

(1) (2, 3), (4, 7) (2) (6, 7), (2, 3) (3) (1, 3), (2, 3) (4) (4, 7), (6, 7)

A-16. The equation, $3x^2 + 4y^2 - 18x + 16y + 43 = C$.

(1) cannot represent a real pair of straight lines for any value of C
(2) represents an ellipse, if $C > 0$
(3) no locus, if $C < 0$
(4) all of these

Se ion (B) : Parametric coordinates, Auxiliary circle, Eccentric angle, Position of a point and Line, Tangent

B-1. The distance of the point 'θ' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from a focus is

(1) $a(e + \cos\theta)$ (2) $a(e - \cos\theta)$ (3) $a(1 + e\cos\theta)$ (4) $a(1 + 2e\cos\theta)$

B-2. Eccentric angle of one of the points on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the centre of the ellipse is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{3}$

B-3. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$ is

(1) outside the ellispe (2) on the ellipse (3) on the major axis (4) on the minor axis

B-4. The tangents at the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its corresponding point Q on the auxiliary circle meet on the line :

(1) $x = a/e$ (2) $x = 0$ (3) $y = 0$ (4) $x = -a/e$

-5. Minimum area of the triangle formed by any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the coordinate axes is

1) 2ab (2) ab (3) $\frac{ab}{2}$ (4) 4ab

B-6. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point P, then eccentric angle of P is
(1) 0 (2) 45° (3) 60° (4) 90°

B-7. The locus of point of intersection of tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose the sum of eccentric angles is constant, is :
(1) a hyperbola (2) an ellipse (3) a circle (4) a straight line

B-8. The equation to the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ included between the co-ordinate axes is the curve :
(1) $9x^2 + 16y^2 = 4x^2y^2$ (2) $16x^2 + 9y^2 = 4x^2y^2$
(3) $3x^2 + 4y^2 = 4x^2y^2$ (4) $9x^2 + 16y^2 = x^2y^2$

B-9. The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$, are
(1) $y = \pm 3$ (2) $x = \pm\sqrt{5}$ (3) $y = 0, y = 6$ (4) none of these

B-10. The curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$, is
(1) ellipse (2) parabola (3) hyperbola (4) circle

B-11. If the line $y = 2x + c$ be a tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$, then c is equal to
(1) ± 4 (2) ± 6 (3) ± 1 (4) ± 8

B-12. If the line $3x + 4y = -\sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$ then, the point of contact is
(1) $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ (2) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ (3) $\left(\frac{1}{\sqrt{7}}, \frac{-1}{\sqrt{7}}\right)$ (4) $\left(\frac{-1}{\sqrt{7}}, \frac{-1}{\sqrt{7}}\right)$

B-13. The equation of tangent to the ellipse $\frac{x^2}{50} + \frac{y^2}{32} = 1$ which passes through a point $(15, -4)$ is
(1) $4x + 5y = 40$ (2) $4x + 35y = 200$ (3) $4x - 5y = 40$ (4) none of these

Section (C) : Normal

C-1. The eccentric angle of the point where the line, $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ is
(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{6}$

C-2. If the line $x\cos\alpha + y\sin\alpha = p$ be normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then
(1) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = a^2 - b^2$ (2) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = (a^2 - b^2)^2$
(3) $p^2(a^2\sec^2\alpha + b^2\text{cosec}^2\alpha) = a^2 - b^2$ (4) $p^2(a^2\sec^2\alpha + b^2\text{cosec}^2\alpha) = (a^2 - b^2)^2$

C-3. The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the positive end of latus rectum is
(1) $x + ey + e^2a = 0$ (2) $x - ey - e^3a = 0$ (3) $x - ey - e^2a = 0$ (4) none of these

C-4. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to
(1) $\frac{2}{3}$ (2) $-\frac{2}{3}$ (3) $\frac{3}{2}$ (4) $-\frac{3}{2}$

C-5. If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one extremity of the minor axis, then the eccentricity of the ellipse is given by the relation

(1) $e^4 + 2e^2 - 4 = 0$ (2) $e^4 + e^2 - 1 = 0$

(3) $e^4 + e^2 - \frac{3}{2} = 0$ (4) $e^4 - e^2 - 1 = 0$

C-6. The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the

(1) $\frac{(a^2 - b^2)\,ab}{a^2 + b^2}$ (2) $\frac{(a^2 + b^2)\,ab}{(a^2 - b^2)}$ (3) $\frac{(a^2 - b^2)}{ab\,(a^2 + b^2)}$ (4) $\frac{a^2 + b^2}{(a^2 - b^2)ab}$

C-7. The value of λ, for which the line $2x - \frac{8}{3}\lambda y = -3$ is a normal to the conic $x^2 + \frac{y^2}{4} = 1$ is

(1) $\pm\frac{\sqrt{3}}{2}$ (2) $\pm\frac{1}{2}$ (3) $-\frac{\sqrt{3}}{4}$ (4) $\pm\frac{3}{8}$

Section (D) : Chord, Pair of Tangents, Chord of Contact, Director Circle, Chord with given mid point and Diameter

D-1. If $\tan\theta_1 \tan\theta_2 = -\frac{a^2}{b^2}$ and the chord joining two points θ_1 and θ_2 on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtend angle α at centre then α =

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) none of these

D-2. The angle between the tangents drawn from the point (2, 2) to the ellipse, $3x^2 + 5y^2 = 15$ is:

(1) $\pi/6$ (2) $\pi/4$ (3) $\pi/3$ (4) $\pi/2$

D-3. The equation of the chord of the ellipse $2x^2 + 5y^2 = 20$ which is bisected at the point (2, 1) is

(1) $4x + 5y + 13 = 0$ (2) $4x + 5y = 13$ (3) $5x + 4y + 13 = 0$ (4) $4x + 5y = 13$

D-4. A triangle ABC right angled at 'A' moves so that its sides touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ all the time. The locus of the point 'A' is

(1) $x^2 + y^2 = 2a^2$ (2) $x^2 + y^2 = 2b^2$ (3) $x^2 + y^2 = a^2 + b^2$ (4) none of these

D-5. The angle between the pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2) is

(1) $\tan^{-1}\left(\frac{12}{5}\right)$ (2) $\tan^{-1}(6\sqrt{5})$ (3) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (4) $\tan^{-1}(12\sqrt{5})$

D-6. If $3x + 4y = 12$ intersect the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at P and Q, then the point of intersection of tangents at P and Q is

(1) (0, 1) (2) (1, –2) (3) $\left(\frac{25}{4}, \frac{16}{3}\right)$ (4) $\left(-\frac{25}{4}, \frac{16}{3}\right)$

D-7. Point/points, from which tangents to the ellipse $5x^2 + 4y^2 = 20$ are perpendicular, is/are :

(1) $(1, 2\sqrt{2})$ (2) $(2\sqrt{2}, 1)$ (3) $(2, \sqrt{5})$ (4) all of these

Section (E) : Properties of Ellipse

E-1. If F_1 & F_2 are the feet of the perpendiculars from the focii S_1 & S_2 of an ellipse $\frac{x^2}{5}+\frac{y^2}{3}=1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to :
(1) 2 (2) 3 (3) 4 (4) 5

E-2. $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ & $\frac{x^2}{4}+\frac{y^2}{b^2}=1$. Then the value of 'b' and the other common tangent are given by :
(1) $b=\sqrt{3}$; $x + 2y + 4 = 0$
(2) $b = 3$; $x + 2y + 4 = 0$
(3) $b=\sqrt{3}$; $x + 2y - 4 = 0$
(4) $b=\sqrt{3}$; $x - 2y - 4 = 0$

E-3. If CF is perpendicular from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent at P, and G is the point where the normal at P meets the major axis, then the product CF · PG is :
(1) a^2 (2) $2b^2$ (3) b^2 (4) $a^2 - b^2$

E-4. If 'P' be a moving point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ in such a way that tangent at 'P' intersect $x=\frac{25}{3}$ at Q then circle on PQ as diameter passes through a fixed point. The fixed point is
(1) (–3, 0) (2) (1, 0) (3) (–1, 0) (4) (3, 0)

E-5. A ray emanating from (6, 2) is incident on ellipse $\frac{(x-1)^2}{45}+\frac{(y-2)^2}{20}=1$ at (4, 6).
The equation of reflected ray (after 1st reflection) is
(1) $x - 2y + 8 = 0$
(2) $x + 2y + 8 = 0$
(3) $x + 2y - 8 = 0$
(4) $x - 2y - 8 = 0$

E-6. P & Q are corresponding points on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, and the auxiliary circle respectively. The normal at P to the ellipse meets CQ in R where C is centre of the ellipse. Then ℓ(CR) is
(1) 5 units (2) 6 units (3) 7 units (4) 8 units

Exercise # 2

Single choice type

1. The length of the common chord of the ellipse $\frac{(x-1)^2}{9}+\frac{(y-1)^2}{4}=1$ and the circle $(x-1)^2 + (y-2)^2 = 1$ is
(1) 2 (2) 4 (3) 8 (4) 0

2. The total number of real common tangents that can be drawn to the ellipse $3x^2 + 5y^2 = 32$ and $25x^2 + 9y^2 = 450$ passing through (3, 5) is
(1) 1 (2) 2 (3) 3 (4) 4

3. The distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$ is
(1) 18 (2) 9 (3) 27 (4) 5

4. If the eccentricity of an ellipse be $\frac{5}{8}$ and the distance between its focii be 10, then its latus rectum is
(1) $\frac{39}{4}$ (2) $\frac{39}{8}$ (3) $\frac{39}{2}$ (4) $\frac{\sqrt{39}}{2}$

Answers

EXERCISE # 1

Section (A) :

A-1. (1) **A-2.** (4) **A-3.** (3) **A-4.** (3)
A-5. (4) **A-6.** (2) **A-7.** (4) **A-8.** (4)
A-9. (3) **A-10.** (2) **A-11.** (2) **A-12.** (2)
A-13. (2) **A-14.** (3) **A-15.** (2) **A-16.** (4)

Section (B) :

B-1. (3) **B-2.** (3) **B-3.** (3) **B-4.** (3)
B-5. (2) **B-6.** (2) **B-7.** (4) **B-8.** (1)
B-9. (3) **B-10.** (1) **B-11.** (2) **B-12.** (4)
B-13. (1)

Section (C) :

C-1. (1) **C-2.** (4) **C-3.** (2) **C-4.** (2)
C-6. (1) **C-7.** (1)

Section (D) :

D-1. (4) **D-2.** (4) **D-3.** (2) **D-4.** (3)
D-5. (3) **D-6.** (3) **D-7.** (4)

Section (E) :

E-1. (2) **E-2.** (1) **E-3.** (3) **E-4.** (4)
E-5. (1) **E-6.** (3)

EXERCISE # 2

1. (4) **2.** (3) **3.** (1) **4.** (1) **5.** (2) **6.** (1) **7.** (3)
8. (1) **9.** (1) **10.** (3) **11.** (1) **12.** (3) **13.** (3) **14.** (1)
15. (4) **16.** (1) **17.** (1) **18.** (3) **19.** (3)

EXERCISE # 3

1. (3) **2.** (2) **3.** (1) **4.** (4) **5.** (1) **6.** (2)
7. (3) **8.** (4) **9.** (2) **10.** (1)

EXERCISE # 4

PART - I

1. (1) **2.** (2) **3.** (2) **4.** (4) **5.** (1) **6.** (4)
7. (1) **8.** (1) **9.** (1) (2)

PART - II

1. (2, 1) **2.** **(i)** D **(ii)** B **3.** C **4.** A

5. $AB = \frac{14}{\sqrt{3}}$, $2x + \sqrt{3}y = 4\sqrt{7}$ **6.** BC

7. D **8.** C **9.** D **10.** C **11.** A

Advance Level Problems

PART - I : OBJECTIVE QUESTIONS

Single choice type :

1. The sum of the squares of the perpendicular on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from two points on the minor axis each at a distance $\sqrt{a^2 - b^2}$ from the centre is

(1) a^2 (2) $\frac{a^2}{2}$ (3) $2a^2$ (4) $4a^2$

2. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then

(1) $b + c = 4a$ (2) $b - c = 2a$
(3) locus of points A is an ellipse (4) locus of point A is a pair of straight lines

3. If α, β are eccentric angles of the extremities of a focal chord of an ellipse, then eccentricity of the ellipse is

(1) $\frac{\cos\alpha + \cos\beta}{\cos(\alpha + \beta)}$ (2) $\frac{\sin\alpha - \sin\beta}{\sin(\alpha - \beta)}$ (3) $\sec\alpha + \sec\beta$ (4) $\frac{\sin\alpha + \sin\beta}{\sin(\alpha + \beta)}$

4. If radii of director circles of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are 2r and r respectively and e_e and e_h be the eccentricities of the ellipse and the hyperbola respectively then

(1) $2e_h^2 - e_e^2 = 6$ (2) $e_e^2 - 4e_h^2 = 6$ (3) $4e_h^2 - e_e^2 = 6$ (4) none of these

5. The distance of the point $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on the ellipse $x^2/6 + y^2/2 = 1$ from the centre of the ellipse is 2, if

(1) $\theta = \pi/3$ (2) $\theta = \pi/6$ (3) $\theta = 5\pi/4$ (4) $\theta = 5\pi/3$

6. The eccentricity of the ellipse which meets the straight line $2x - 3y = 6$ on the x-axis and the straight line $4x + 5y = 20$ on the y-axis and whose axes lie along the coordinates axes, is

(1) $\frac{1}{2}$ (2) $\frac{4}{5}$ (3) $\frac{\sqrt{3}}{4}$ (4) $\frac{\sqrt{7}}{4}$

7. The distance of the point of contact from the origin of the line $y = x - \sqrt{7}$ with the ellipse $3x^2 + 4y^2 = 12$, is

(1) $\sqrt{3}$ (2) 2 (3) $\frac{5}{\sqrt{7}}$ (4) none of these

8. The locus of extremities of latus rectum of the family of ellipse $b^2x^2 + y^2 = a^2b^2$ where b is a parameter ($b^2 < 1$), is-

(1) $x^2 \pm a^2y^2 = a^2$ (2) $x^2 \pm ay = a^2$ (3) $x \pm ay^2 = a^2$ (4) none of these

9. A series of concentric ellipses $E_1, E_2, \ldots\ldots, E_n$ are drawn such that E_n touches the extremities of the major axis of E_{n-1} and the foci of E_n coincide with the extremities of minor axis of E_{n-1}. If the eccentricity of the ellipses is independent of n, then the value of the eccentricity, is

(1) $\frac{\sqrt{5}}{3}$ (2) $\frac{\sqrt{5}-1}{2}$ (3) $\frac{\sqrt{5}+1}{2}$ (4) $\frac{1}{\sqrt{5}}$

10. The set of values of 'a' for which $(13x - 1)^2 + (13y - 2)^2 = a(5x + 12y - 1)^2$ represents an ellipse is

(1) $1 < a < 2$ (2) $0 < a < 1$ (3) $3 < a < 4$ (4) none of these

11. If (5, 12) and (24, 7) are the focii of a conic passing through the origin then the eccentricity of conic is

(1) $\frac{\sqrt{386}}{38}$ (2) $\frac{\sqrt{386}}{6}$ (3) $\frac{\sqrt{386}}{13}$ (4) $\frac{\sqrt{386}}{25}$

Answers

(3) 2. (3) 3. (4) 4. (3) 5. (3) 6. (4) 7. (3)

(2) 9. (2) 10. (2) 11. (1)

1. $y = mx + \sqrt{a^2m^2+b^2}$

Distance $p_1^2 + p_2^2 = \left(\dfrac{\sqrt{a^2m^2+b^2}+\sqrt{a^2-b^2}}{\sqrt{1+m^2}}\right)^2 + \left(\dfrac{\sqrt{a^2m^2+b^2}-\sqrt{a^2-b^2}}{\sqrt{1+m^2}}\right)^2$

$= \dfrac{2(a^2m^2+b^2+a^2-b^2)}{1+m^2} = 2a^2$

2. $\cos B + \cos C = 4\sin^2\dfrac{A}{2}$

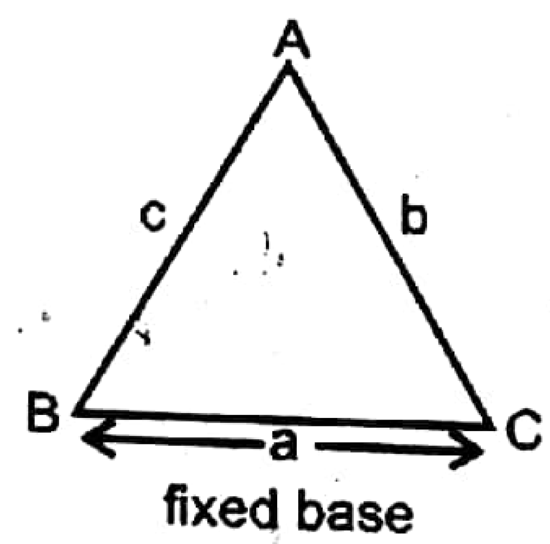
$\Rightarrow \quad 2\cos\dfrac{B+C}{2}\cos\dfrac{B-C}{2} = 4\sin^2\dfrac{A}{2}$

$\Rightarrow \quad 2\sin\dfrac{A}{2}\left[\cos\dfrac{B-C}{2} - 2\sin\dfrac{A}{2}\right] = 0$

$\Rightarrow \quad \cos\left(\dfrac{B-C}{2}\right) - 2\cos\left(\dfrac{B+C}{2}\right) = 0 \qquad$ as $\sin\dfrac{A}{2} \neq 0$

$\Rightarrow \quad -\cos\dfrac{B}{2}\cos\dfrac{C}{2} + 3\sin\dfrac{B}{2}\sin\dfrac{C}{2} = 0$

$\Rightarrow \quad \tan\dfrac{B}{2}\tan\dfrac{C}{2} = \dfrac{1}{3}$



$\Rightarrow \quad \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}\cdot\dfrac{(s-b)(s-a)}{s(s-c)}} = \dfrac{1}{3}$

$\Rightarrow \quad \dfrac{s-a}{s} = \dfrac{1}{3} \quad \Rightarrow \quad 2s = 3a \quad \Rightarrow \quad b + c = 2a$

$\therefore$ Locus of A is an ellipse

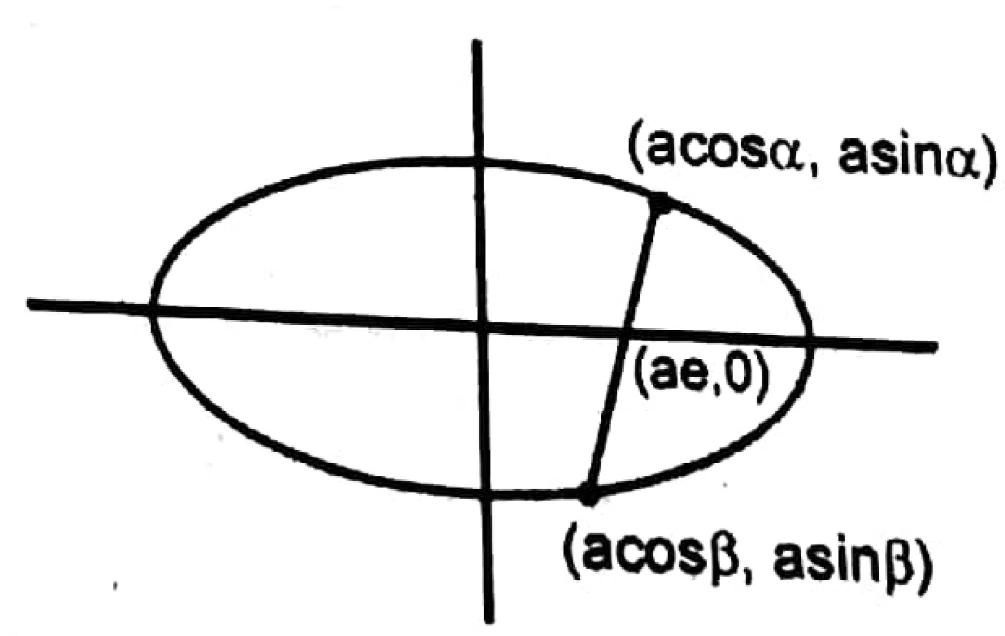
3. Let equation of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad (a > b)$

Here chord is given by

$\dfrac{x}{a}\cos\left(\dfrac{\alpha+\beta}{2}\right) + \dfrac{y}{b}\sin\left(\dfrac{\alpha+\beta}{2}\right) = \cos\left(\dfrac{\alpha-\beta}{2}\right)$

it passes through (ae, 0)

$\therefore \; e\cos\left(\dfrac{\alpha+\beta}{2}\right) = \cos\left(\dfrac{\alpha-\beta}{2}\right)$



$e = \dfrac{2\cos\left(\dfrac{\alpha}{2}-\dfrac{\beta}{2}\right)}{2\cos\left(\dfrac{\alpha}{2}+\dfrac{\beta}{2}\right)} \times \dfrac{\sin\left(\dfrac{\alpha+\beta}{2}\right)}{\sin\left(\dfrac{\alpha+\beta}{2}\right)} = \dfrac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$

4. Equation of director circles of ellipse and hyperbola are respectively.

$x^2 + y^2 = a^2 + b^2$

and $x^2 + y^2 = a^2 - b^2$

$a^2 + b^2 = 4r^2$ ......(1)

$a^2 - b^2 = r^2$ ......(2)

So $2a^2 = 5r^2$

$$a^2 = \frac{5r^2}{2}$$

$$b^2 = 4r^2 - \frac{5r^2}{2}$$

$$b^2 = \frac{3r^2}{2}$$

$\because \quad e_e^2 = 1 - \frac{b^2}{a^2} \quad \Rightarrow \quad e_e^2 = 1 - \frac{3r^2}{2} \times \frac{2}{5r^2} = 1 - \frac{3}{5} = \frac{2}{5}$

$\because \quad e_h^2 = 1 + \frac{b^2}{a^2} \quad \Rightarrow \quad e_h^2 = 1 + \frac{3}{5} = \frac{8}{5}$

So $4e_h^2 - e_e^2 = 4 \times \frac{8}{5} - \frac{2}{5} = \frac{30}{5} = 6$

5. $6\cos^2\theta + 2\sin^2\theta = 4$

$2 + 4\cos^2\theta = 4$

$4\cos^2\theta = 2$

$\cos\theta = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}, n \in I$

So $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

6. The straight line $2x - 3y = 6$ meets the x-axis at $A \equiv (3, 0)$ and the straight line $4x + 5y = 20$ meets the y-axis at $B \equiv (0, 4)$

Thus, for the given ellipse, we have

eccentricity $e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$.

7. Equation of the given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of a tangent to the ellipse at any point (h, k) on the ellipse, is $\frac{xh}{4} + \frac{yk}{3} = 1$

Comparing it with the equation of the given tangent, we have

$$\frac{h}{4} = \frac{-k}{3} = \left(\frac{3h^2 + 4k^2}{3.4^2 + 4.3^2}\right)^{1/2} = \left(\frac{12}{84}\right)^{1/2}$$

[$\because$ (h, k) lies on the ellipse $\therefore$ $3h^2 + 4k^2 = 12$]

gives $h = 4/\sqrt{7}$ and $-3/\sqrt{7}$

So distance $= \sqrt{\left(\frac{4}{\sqrt{7}}\right)^2 + \left(\frac{3}{\sqrt{7}}\right)^2} = \frac{5}{\sqrt{7}}$

8. The given ellipse $\frac{x^2}{a^2} + \frac{y^2}{a^2b^2} = 1$

has eccentricity $e = \sqrt{1-b^2}$

Thus, if P(h, k) be the ends of the latus rectum of the ellipse, then we have

$$h = \pm ae = \pm a\sqrt{1-b^2} \qquad ....(1)$$

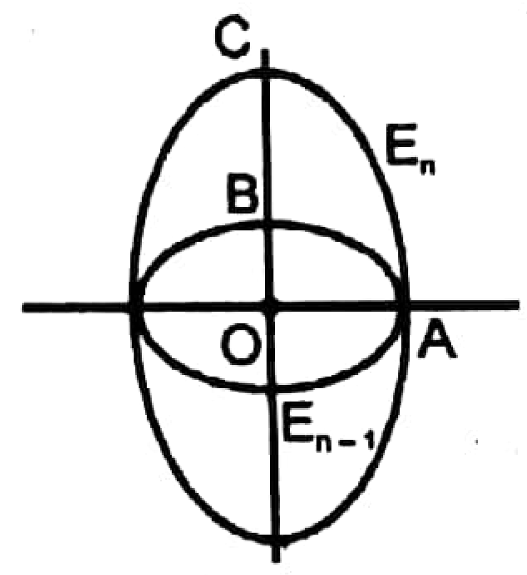
and $$k = \pm \frac{a^2b^2}{a} = \pm ab^2 \qquad ....(2)$$

Eliminating b from equations (1) and (2), we have $1 - \frac{h^2}{a^2} = \pm \frac{k}{a}$

i.e. $h^2 \pm ak = a^2$

Hence, equation of the required locus is $x^2 \pm ay = a^2$

9. The figure shows two ellipses E_{n-1} and E_n.
The eccentricity is given to be independent of n, implies that the ratio of minor axis to the major axis, is same for all the ellipses.



For ellipse E_{n-1}, let
minor axis = b, major axis = a
For ellipse E_n, we have

minor axis = a, major axis = $\frac{OB}{e} = \frac{b}{e}$

[$\because$ B is the focus of E_n]
assuming e to be the eccentricity. Thus, we have

$$\frac{b}{a} = \frac{a}{b/e}$$

$\Rightarrow$ $e = \frac{b^2}{a^2} = 1 - e^2$ $\Rightarrow$ $e^2 + e - 1 = 0$

gives $e = \frac{\sqrt{5}-1}{2}$ [$\because$ e must be + ve]

10. $$\left(x - \frac{1}{13}\right)^2 + \left(y - \frac{2}{13}\right)^2 = a\left[\frac{5x+12y-1}{13}\right]^2$$

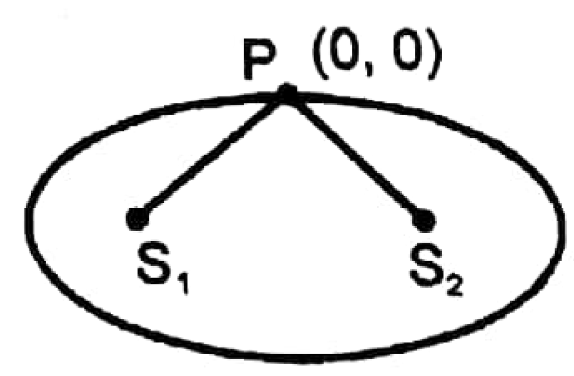
$PS^2 = e^2\ PM^2$

here $a = e^2$

$0 < e < 1$ for ellipse

so $0 < a < 1$

11. we know that for ellipse $PS_1 + PS_2 = 2a$



So $13 + 25 = 2a$

$2a = 38$

and $S_1 S_2 = 2ae = \sqrt{386}$

So $e = \frac{\sqrt{386}}{38}$

and for hyperbola $|PS_1 - PS_2| = 2a$

$2a = |13 - 25|$

So $2a = 12$

and $S_1 S_2 = 2ae = \sqrt{386}$

So $e = \frac{\sqrt{386}}{12}$

SOLUTIONS

EXERCISE # 1

Section (A) :

A-1. $\sqrt{(x-1)^2+(y+1)^2} = \frac{1}{2}\left|\frac{x-y-3}{\sqrt{1^2+1^2}}\right|$

Squaring, we have
$7x^2 + 7y^2 + 7 - 10x + 10y + 2xy = 0$

A-4. $9x^2 + 4y^2 = 1$

$\Rightarrow \frac{x}{1/9} + \frac{y^2}{1/4} = 1$ $\Rightarrow$ Length of latusrectun = $\frac{2a^2}{b} = \frac{4}{9}$

A-6. Max. area = $\frac{1}{2} \times 2ae \times b = \frac{1}{2} \times 2 \times 3 \times 4 = 12$

A-10. $\frac{x^2}{r-2} + \frac{y^2}{5-r} = 1$ For ellipse
$2 < r < 5$
$2 < r < 5$

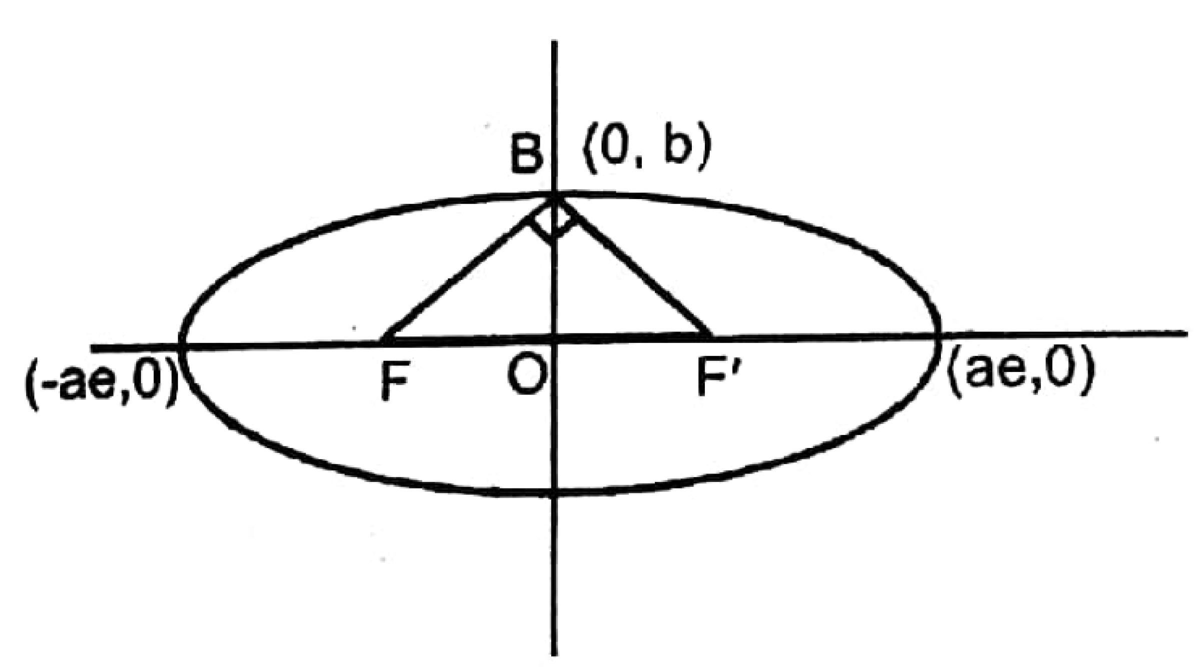
A-12. Use the relation between a, b and e
Given $\angle FBF' = 90°$

$\therefore \frac{b}{-ae} \times \frac{b}{ae} = -1$

$\Rightarrow b^2 = a^2e^2$

$\Rightarrow a^2(1 - e^2) = a^2e^2$

$\therefore e = \frac{1}{\sqrt{2}}$



A-13. $2ae = 10$ $\frac{a}{e} - ae = 15$

$ae = 5$ $\frac{5}{e^2} - 5 = 15$ $\Rightarrow \frac{5}{e^2} = 20$ $\Rightarrow e = \frac{1}{2}$

A-16. $3(x-3)^2 + 4(y+2)^2 = C$
if $C = 0$ a point
if $C > 0$ ellipse
if $C < 0$ no locus.

Section (B) :

B-3. $4(x^2 - 4x + 4) + 9(y^2 - 64 + 9) = 36$
$4(x-2)^2 + 9(y-3)^2 = 36$

$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1.$

Equation of major axis y = 3.
Equation of minor axis x = 2

B-4. $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$ ......(1)

$x^2 + y^2 = a^2$

$ax\cos\phi + ay\sin\phi = a^2$

$x\cos\phi + y\sin\phi = a$

$\frac{x}{a}\cos\phi + \frac{y}{a}\sin\phi = 1$ ......(2)

Solving (1) and (2) $y = 0$

B-6. Let eccentric angle be θ, then equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ ...(1)

given equation is

$\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ ...(2)

comparing (1) and (2)

$\cos\theta = \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$

B-8. Let tangent $\frac{x}{4}\cos\theta + \frac{y}{3}\sin\theta = 1$

$A \equiv (4\sec\theta, 0)$ $B \equiv (0, 3\,\text{cosec}\theta)$

Let mid point be (h, k) ° $\left(2\sec\theta, \frac{3\,\text{cosec}\theta}{2}\right)$

$\cos\theta = \frac{2}{h}, \sin\theta = \frac{3}{2k}$

$\frac{4}{x^2} + \frac{9}{4y^2} = 1 \Rightarrow 16y^2 + 9x^2 = 4x^2y^2$

B-13. $y + 4 = m(x - 15)$

$y = mx - (15m + 4)$ .... (i)

for tangent

$(15m + 4)^2 = 50m^2 + 32$

$175m^2 + 120m - 16 = 0$

$(35m - 4)(5m + 4) = 0$

$m = \frac{4}{35}, \frac{-4}{5}$

putting in (i)

$4x + 5y = 40$

$4x - 35y = 200$

Section (C) :

C-2. Equation of normal

$ax\sec\phi - by\,\text{cosec}\phi = a^2 - b^2$ ...(1)

$x\cos\alpha + 4\sin\alpha = p$ ...(2)

$\frac{a\sec\phi}{\cos\alpha} = \frac{-by\,\text{cosec}\phi}{\sin\alpha} = \frac{a^2 - b^2}{p}$

$\Rightarrow \cos\phi = \frac{ap}{(a^2 - b^2)} \times \sec\alpha$ ...(3)

Our Pan India Network

Resonance Eduventures Pvt. Ltd.

CORPORATE OFFICE: J-2, Jawahar Nagar Main Road, Kota (Rajasthan) - 324005

Tel.: 0744-3192222, 3012222, 3022222, 2437144 | **Fax:** 0744-2427144, 022-39167222

Call @ Study Centres (SCs): Jaipur, Bhopal, Lucknow, Nagpur, Udaipur, Bhubaneswar, Patna, Jodhpur, Ajmer: (STD Code) 3192222
Visakhapatnam : 0891-2500005 | Delhi, Kolkata, Mumbai, Bengaluru, Ahmedabad: (STD Code) 31922222
Call @ Information Centres (ICs): Chandigarh, Indore, Ranchi: (STD Code) 3192222

To know more SMS RESO at 56677 I Toll Free No.: 1800 200 2244

contact@resonance.ac.in | www.resonance.ac.in | WAPsite: m.resonance.ac.in