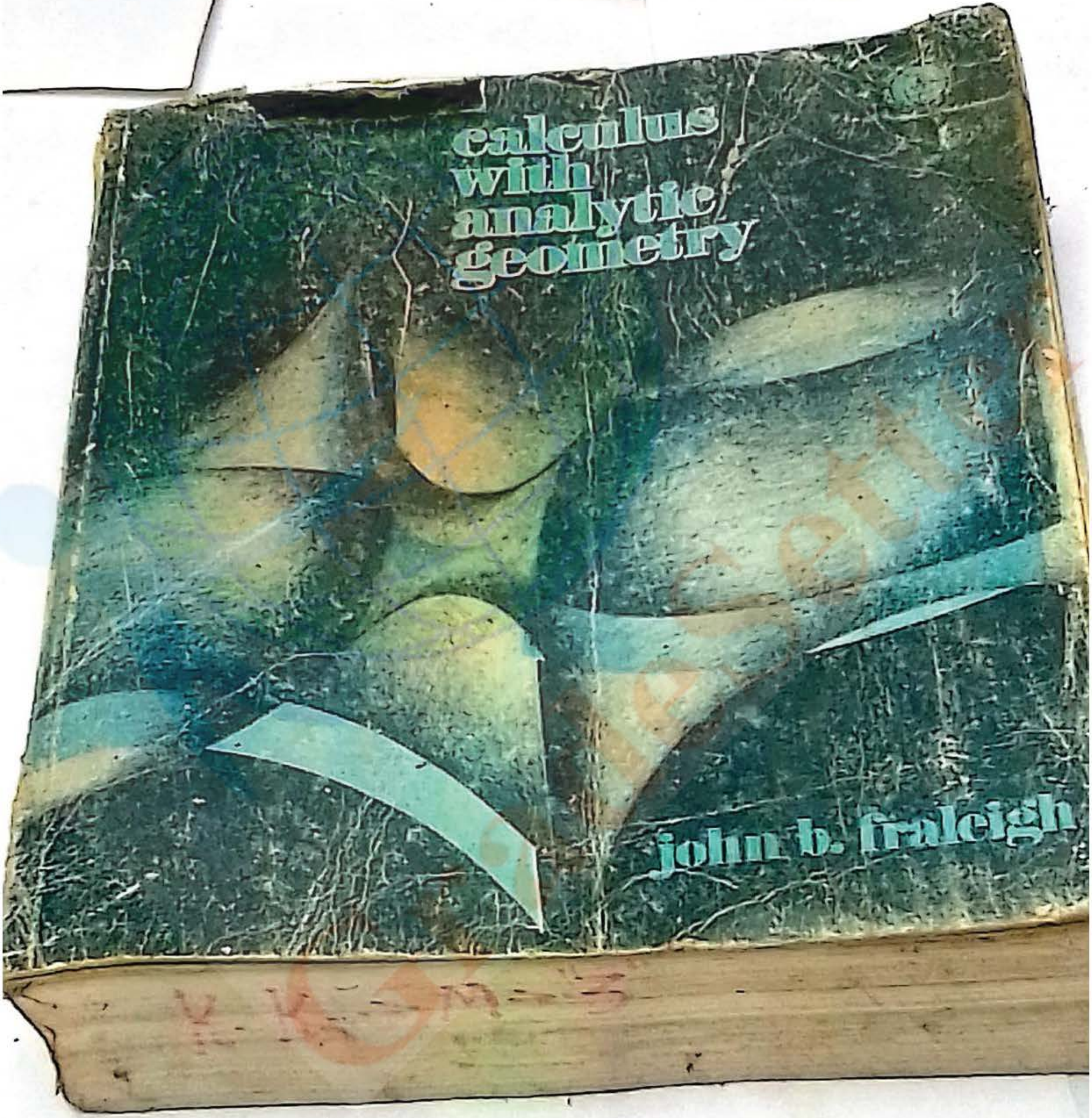

calculus
with
analytic
geometry

john b. fraleigh

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John B. Fraleigh
University of Rhode Island

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Reading, Massachusetts • Amsterdam • London
Manila • Singapore • Sydney • Tokyo

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This book is in the
ADDISON-WESLEY SERIES IN MATHEMATICS

Consulting editor:
Lynn H. Loomis

Sponsoring editor: Steve Quigley
Production editor: Mary Cafarella
Designer: Katrine Stevens
Illustrator: Carmela Ciampa
Cover design: Ann Scrimgeour

ISBN 0-201-10488-1
ABCDEFGHIJKL-KSL-898765432

preface

This text is designed for a standard college calculus sequence for students in the physical or social sciences. Such a sequence typically spans three semesters or four quarters. Students are expected to have a background of high school algebra and geometry.

The college calculus sequence contains a great deal of very important mathematics, which students actually use after the course is completed. Few mathematics courses present so much new material at such a rapid pace: this poses a real challenge for the instructor. There is seldom enough time in the classroom to provide complete coverage and supervised drill. Accordingly, students must depend on the text for ideas as well as for exercises.

In this text, I have made every effort to present calculus as clearly and intuitively as possible. Subtle points and proofs of difficult theorems have been omitted. Emphasis is on development of an intuitive but accurate feeling for the subject, and on secure technical competence. The features mentioned below are typical of my efforts to meet this challenge and to provide a really useful text for students.

I introduce the derivative promptly, as opposed to some texts that spend the first hundred pages on preliminaries. So much calculus has to be learned that we must get right to work on it.

A summary at the end of each *lesson* identifies and collects the most important ideas and formulas. This makes the text exceptionally easy to use for study and review.

Reading mathematics is an art that is very different from reading a novel. Students need practice in reading mathematics, and instructors should encourage and require them to do so. All too often calculus students are tested only on material that has been thoroughly covered in the classroom. Such a practice leads students to feel that independent study from a mathematics text is impossibly difficult. Each semester, I like to assign my classes at least three text lessons for independent reading, problem solving, and testing, with no classroom coverage. The text contains several sections that may be assigned out of the sequence in which they appear, and are thus ideal for independent study. I recommend the following:

Semester 1
Section 1.5 Graphs of monomial and quadratic functions
Section 5.2 Newton's method (includes the intermediate-value theorem)
Section 6.5 Numerical methods of integration

Semester 2
Section 8.6 The hyperbolic functions
Section 9.4 Integration of rational functions of sin x and cos x
Section 12.2 Synthetic definitions of conic sections

Semester 3
Section 14.2 Quadric surfaces
Section 16.6 Differentiation of implicit functions (several variables)
Section 17.2 Lagrange multipliers

Assigning these sections for independent study allows more classroom time for basic concepts.

In place of the usual collection of miscellaneous exercises at the end of a chapter, there are two sets of review exercises, followed by a set of more challenging exercises. Each review exercise set gives students an easy way to test their mastery of basic material, and to determine areas that need more study.

Suggested step-by-step procedures are given for solving certain types of problems that cause many students difficulty, such as related rate and maximum-minimum word problems.

Calculus of the trigonometric functions appears in the first-semester portion of the text, shortly after the chain rule. Prompt introduction of this topic helps students understand and remember the chain rule. Two review lessons on the trigonometric functions are supplied for students who need them.

I feel that the use of numerical methods gives a concrete understanding and appreciation of the notions of calculus. Accordingly, the text has more emphasis on numerical methods than most. In particular, there are optional calculator exercises, designed to illustrate concepts of calculus as well as to emphasize numerical techniques.

Some instructors, myself included, begrudge the amount of time often spent on conic sections, since so much calculus must be covered in so short a time. The material in Chapter 12 is arranged so that only one lesson (on sketching) need be spent on conic sections in order to do the remaining unstarred material in the text. All the usual material on conic sections is included for those instructors who do wish to cover it.

The first semester of the calculus sequence presents powerful ideas and techniques, solving problems that students were previously unable to attack. This first semester is the most exciting part of the sequence and, indeed, is one of the most exciting semesters of undergraduate mathematics. The second semester is often a letdown, using the ideas of the first semester with more functions and different coordinate systems, and developing integration technique. I like to have at least one major, exciting topic in the second semester, so I am placing series in the middle of the text. The first series chapter (Chapter 10) is exceptional in training students to determine convergence or divergence of series at a glance, as a mathematician would, based on rigorous tests but without always writing them out. Of course, series can be left as the last topic of the sequence if the instructor prefers.

Some computer graphics are included, but each appears only as a companion to an artist's sketch. Pencil and paper are still the basic tools for studying mathematics. It is important that students develop some ability in sketching to strengthen their geometric intuition. A computer-generated picture, with its myriad precise curves, is ordinarily impossible for students to reproduce. Good pedagogy requires including a sketch by an artist, whom students may emulate. I worked out the computer graphics and programs at the URI Computer Center, where the staff was very helpful.

A Student Supplement is available. The supplement goes through the text, lesson by lesson, warning students of mistakes often made, and then giving complete solutions of every third problem. A Solutions Manual, which works out solutions to all problems, is available for the instructor. I myself prepared these manuals, as well as the answers to odd-numbered exercises at the end of the text. Consequently, I take full responsibility for mistakes; I hope their number is small.

I am indebted to the reviewers of the manuscript for their many valuable suggestions. Some read the manuscript with great care at two stages of development. Among the reviewers were James E. Arnold, Jr. (University of Wisconsin at Milwaukee), Ross A. Beaumont (University of Washington), Arthur T. Copeland (University of New Hampshire), William R. Fuller (Purdue University), Kendell Hyde (Weber State College), and Joan H. McCarter (Arizona State University).

I especially thank Steve Quigley, mathematics editor, and Lynn Loomis, consulting editor, of Addison-Wesley for their advice, encouragement, time, and patience during the entire project.

J. B. F.

Kingston, R.I.
November, 1979

contents

4 The Trigonometric Functions 97

5 Applications of the Derivative 113

6 The Integral 153

7 Applications of the Integral 191

8 Other Elementary Functions 235

functions and graphs

1

1.1 COORDINATES AND DISTANCE

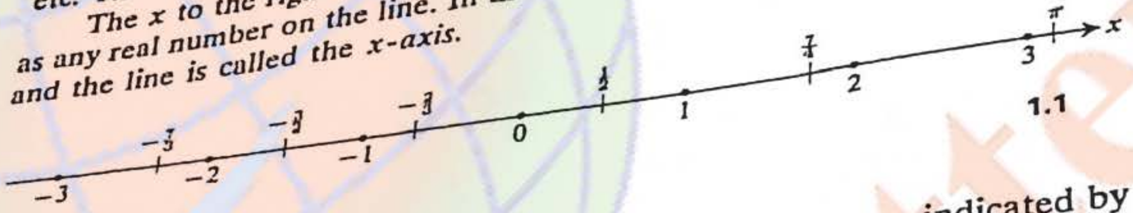
1.1.1 Coordinates on the line

A **real number** is one that can be written as an unending decimal, positive or negative, or zero. For example, $3 = 3.000000\ldots$, $-\frac{2}{3} = -0.666666\ldots$, and $\pi = 3.141592\ldots$ are real numbers. It is very useful to visualize real numbers as points on the *number line*. Take a line extending infinitely in both directions. Using the real numbers, you can make this line into an infinite ruler (see Fig. 1.1). Label any point on the line with 0 and any point to the right of 0 with 1; this fixes the scale. Each positive real number r corresponds to the point a distance r units to the right of 0, while a negative number $-s$ corresponds to the point a distance s units to the left of 0. The arrow on the line indicates the positive direction. For real numbers r and s, the notation $r < s$ (read "*r is less than s*") means that r is to the left of s on the number line. For example,

$$-\tfrac{2}{3} < \tfrac{1}{2}, \qquad 2 < \pi, \qquad -3 < -\tfrac{7}{3},$$

etc. The notation $r \leq s$ is read "*r is less than or equal to s.*"

The x to the right of the arrow in Fig. 1.1 indicates that you think of x as any real number on the line. In this context, x is known as a *real variable*, and the line is called the *x-axis*.



1.1

Example 1 The points x on the line that satisfy the relation $0 \leq x \leq 2$ are indicated by the heavy line and the dark points in Fig. 1.2. Both 0 and 2 satisfy this relation. ‖

1.2

1.3

Example 2 The points satisfying $-1 < x \leq 1$ are indicated by the heavy line together with the dark point 1 in Fig. 1.3. This time -1 does not satisfy the relation, while 1 does. ‖

The collection of points x satisfying a relation of the form $a \leq x \leq b$ will be important in calculus. This set of points is the **closed interval** $[a, b]$. The adjective "closed" is used to indicate that both endpoints, a and b, are considered part of the interval; that is, the doors are "closed" at both ends of the interval by these points.

The distance from a point r to the point 0 is known as the **absolute value** of the number r and is denoted by $|r|$. For example,

$$|5| = |-5| = 5,$$

for both 5 and −5 are five units from 0. Consequently,

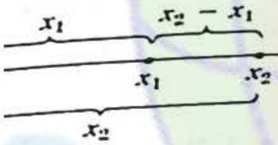
$$|r| = r \quad \text{for any positive number } r,$$

while

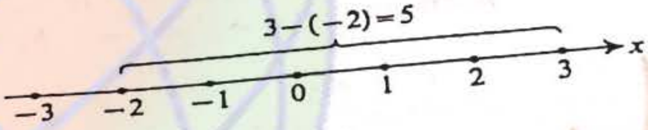
$$|-s| = s \quad \text{for any negative number } -s.$$

Of course, $|0| = 0$.

Consider now the distance between any two points on the line. It is convenient to use subscripted notation x_1 and x_2 for individual numbers on the x-axis, although their values are not specifically given. The distance between the points x_1 and x_2, shown in Fig. 1.4, is surely $x_2 - x_1$. You can easily convince yourself that, for any two points x_1 and x_2, where $x_1 \leq x_2$, the distance between them is $x_2 - x_1$.



1.4



1.5

Example 3 The distance between −2 and 3 is $3 - (-2) = 5$, as indicated in Fig. 1.5. ‖

Distance on the line

Now for *any* points x_1 and x_2, the distance between them is either $x_1 - x_2$, or $x_2 - x_1$, whichever is nonnegative. This nonnegative magnitude is, of course, $|x_2 - x_1|$. Thus the distance from 3 to −2 is $|(-2) - 3| =$ $|-5| = 5$. Another way of expressing this nonnegative difference $\sqrt{(x_2 - x_1)^2}$, where the square root symbol $\sqrt{\ }$ always yields the *nonnegati* square root of the number. Later in this section, you will see that this squa root expression extends naturally to a formula for the distance between t points in a plane.

Example 4 For the points −2 and 3,

$$\sqrt{(3-(-2))^2} = \sqrt{5^2} = \sqrt{25} = 5;$$

and also,

$$\sqrt{((-2)-3)^2} = \sqrt{(-5)^2} = \sqrt{25} = 5. \quad \|$$

Exercise 4 asks you to show that $(a + b)/2$ is the same distance as from b, so that $(a + b)/2$ is the *midpoint of* $[a, b]$.

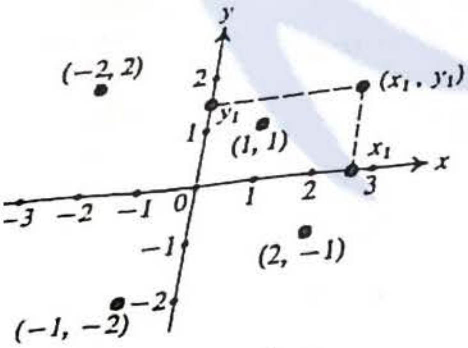
elta notation

Often you need to know not only the distance from x_1 to whether x_1 is to the left or right of x_2. The change $x_2 - x_1$ in x-valu from x_1 to x_2 (in that order) is positive if $x_1 < x_2$ and negative if Very soon, in calculus you will want to let Δx (read "*delta x*") b positive or negative change in x-value. It is a good idea to start right now so that you get used to this delta notation. Think, geometrically, of $\Delta x = x_2 - x_1$ as the *signed* length of the *directed* line segment *from* x_1 *to* x_2.

1.1.2 Coordinates in the plane

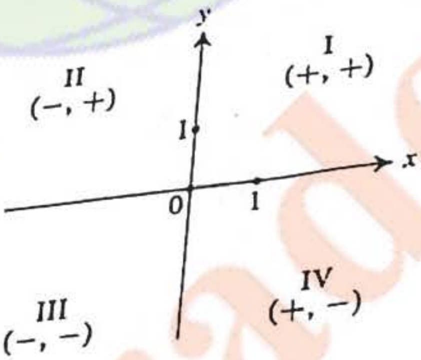
Take two copies of the number line (with equal scales) and place them perpendicular to each other in a plane, so that they intersect at the point 0 on each line (see Fig. 1.6). With each point in the plane, we associate an ordered pair (x_1, y_1) of numbers, as follows: The first number x_1 gives the left-right position of the point according to the location of x_1 on the horizontal number line. Similarly, the second number y_1 gives the up-down position of the point according to the location of y_1 on the vertical number line (see Fig. 1.6). Conversely, given any ordered pair of numbers such as (2, −1), there is a unique point in our plane associated with it.

The solid lines of Fig. 1.6 are the **coordinate axes**. In particular, the horizontal axis is the **x-axis** and the vertical axis the **y-axis**, according to the labels at the arrows. For the point (x_1, y_1), the number x_1 is the **x-coordinate** of the point, while y_1 is the **y-coordinate**. The coordinate axes naturally divide the plane into four pieces or **quadrants**, according to the signs of the coordinates of the points. The quadrants are usually numbered as shown in Fig. 1.7. The point (0, 0) is the origin. This introduction of coordinates allows you to use numbers and their arithmetic as a tool in studying geometry. The term *analytic geometry* is used for the study of geometry using coordinates. Of equal importance, coordinatization allows you to draw geometric pictures illustrating a great deal of numerical work.

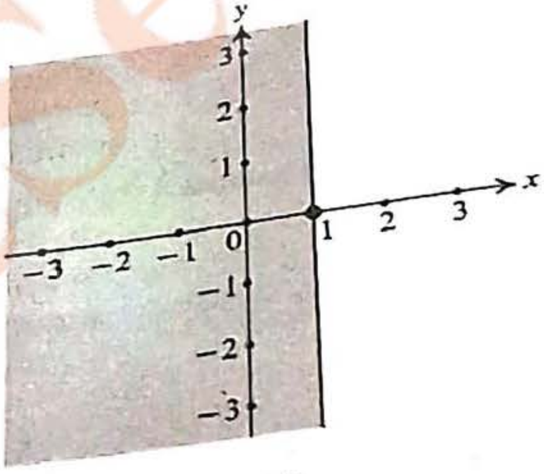


The Euclidean plane

1.6



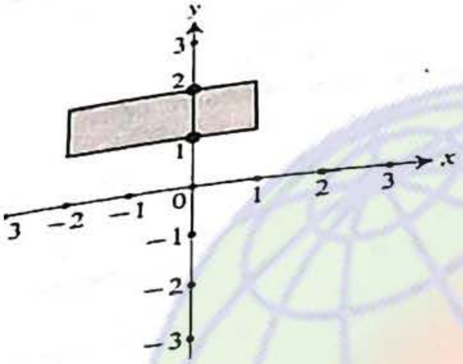
1.7



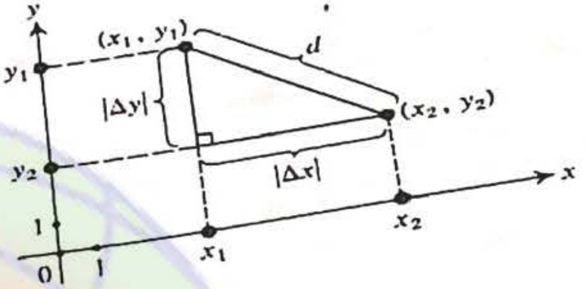
1.8

Example 5 The portion of the plane consisting of those points (x, y) satisfying th relation $x \le 1$ is shown in Fig. 1.8. ‖

Example 6 The portion of the plane consisting of the points (x, y) satisfying *both* $-2 \leq x \leq 1$ and $1 \leq y \leq 2$ is shown in Fig. 1.9. ‖



1.9



1.10

Finally, let's find the distance between two points (x_1, y_1) and $(x_2, y$ the plane. Referring to Fig. 1.10, let $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y$ that $|\Delta x|$ and $|\Delta y|$ are the lengths of the legs of the right triangle shov the figure. The distance between (x_1, y_1) and (x_2, y_2) is the length d hypotenuse of this triangle; so, by the Pythagorean theorem,

$$d^2 = |\Delta x|^2 + |\Delta y|^2.$$

Distance in the plane Since the terms in (1) are squared, the absolute-value symbols needed, so that $d^2 = (\Delta x)^2 + (\Delta y)^2$ and

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 7 The distance between $(2, -3)$ and $(-1, 1)$ is

$$\sqrt{(-1 - 2)^2 + (1 - (-3))^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} =$$

The availability of inexpensive electronic calculators (the models with memory) makes the computation of (2) easy. The e follow conclude with a calculator portion.

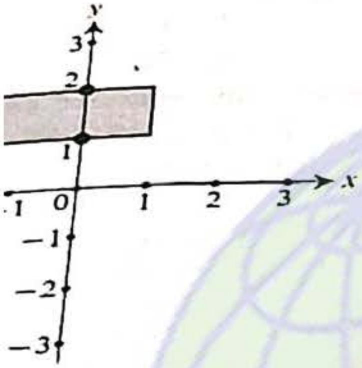
SUMMARY

1. *The closed interval* $[a, b]$ *consists of all points* x *such that*
2. *The distance from* x_1 *to* x_2 *on the line is* $|x_2 - x_1| = \sqrt{(x_2}$
3. *The signed length of the directed line segment from* x_1 *to*

   $\Delta x = x_2 - x_1 =$ *(Number where you stop) − (Number wh*
4. *The midpoint of* $[a, b]$ *is* $(a + b)/2$.
5. *The distance between* (x_1, y_1) *and* (x_2, y_2) *in the plane*

   $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 6 The portion of the plane consisting of the points (x, y) satisfying *both* $-2 \le x \le 1$ and $1 \le y \le 2$ is shown in Fig. 1.9. ‖



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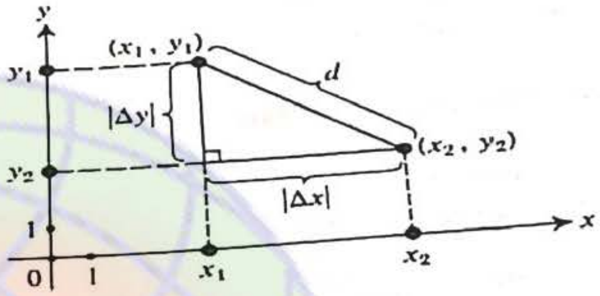

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MMARY

1. *The closed interval $[a, b]$ consists of all points x such that $a \le$*
2. *The distance from x_1 to x_2 on the line is $|x_2 - x_1| = \sqrt{(x_2 - x_1}$*
3. *The signed length of the directed line segment from x_1 to x_2 is*

   $\Delta x = x_2 - x_1 =$ *(Number where you stop) − (Number where yc*
4. *The midpoint of $[a, b]$ is $(a + b)/2$.*
5. *The distance between (x_1, y_1) and (x_2, y_2) in the plane is*

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXERCISES

1. Sketch, as in Fig. 1.2 and Fig. 1.3, all points x (if there are any), that satisfy the given relation.
   a) $2 \leq x \leq 3$ b) $x \leq 0$
   c) $x^2 = 4$ d) $x^2 < 4$
   e) $x^2 \leq 4$ f) $5 \leq x \leq -1$
2. Find the distance between the given points on the line.
   a) 2 and 5 b) −1 and 4 c) −3 and −6
3. Find the distance between the given points on the line.
   a) $-\frac{5}{2}$ and 12 b) $-\frac{8}{3}$ and $-\frac{14}{3}$
   c) $\sqrt{2}$ and $-2\sqrt{2}$ d) $\sqrt{2}$ and π
4. Show that, for any a and b on the line, the distance from $(a + b)/2$ to a is the same as the distance from $(a + b)/2$ to b.
5. Find the midpoint of each of the following intervals.
   a) $[-1, 1]$ b) $[-1, 4]$ c) $[-6, -3]$
   d) $[-\frac{1}{2}, \frac{7}{3}]$ e) $[-2\sqrt{2}, \sqrt{2}]$ f) $[\sqrt{2}, \pi]$
6. Find the signed length Δx of the directed line segment
   a) from 2 to 5 b) from 3 to −7
   c) from −8 to −1 d) from 10 to 2
7. Sketch the points (x, y) in the plane satisfying the indicated relations, as in Examples 5 and 6.
   a) $x = 1$
   b) $-1 \leq x \leq 2$
   c) $x = -1$ and $-2 \leq y \leq 3$
   d) $x = y$ and $-1 \leq x \leq 1$
8. Proceed as in Exercise 7.
   a) $x \leq y$ b) $x = -y$
   c) $y = 2x$ d) $2x \geq y$
9. Find the cordinates of the indicated point.
   a) The point such that the line segment joining it to $(2, -1)$ has the x-axis as perpendicular bisector
   b) The point such that the line segment joining it to $(-3, 2)$ has the y-axis as perpendicular bisector
   c) The point such that the line segment joining it to $(-1, 3)$ has the origin as midpoint
   d) The point such that the line segment joining it to $(2, -4)$ has $(2, 1)$ as midpoint
10. Find the distance between the given points.
    a) $(-2, 5)$ and $(1, 1)$ b) $(2, -3)$ and $(-3, 5)$
    c) $(2\sqrt{2}, -3)$ and $(-\sqrt{2}, 2)$
    d) $(2\sqrt{3}, 5\sqrt{7})$ and $(-4\sqrt{3}, 2\sqrt{7})$
11. To reach the Edwards' home from the center of town, you drive two miles due east on Route 37 and then five miles due north on Route 101. Assuming that the surface of the earth near town is approximately flat, find the distance, as the crow flies, from the center of town to the Edwards' home.
12. Refer to Exercise 11; suppose you drive six miles due west on Route 37 and then four miles due south on Route 43 to reach the Hammonds' house from the center of town. Find the distance from the Edwards' home to the Hammonds' as the crow flies.

calculator exercises

13. Find the midpoint of $[-2\sqrt{3}, 5\sqrt{7}]$.
14. Find the signed length of the directed line segment from $22\sqrt{2}$ to π^3.
5. Find the distance between $(2, -3)$ and $(4, 1)$.
16. Find the distance between $(-3.7, 4.23)$ and $(8.61, 7.819)$.
17. Find the distance between $(\pi, -\sqrt{3})$ and $(8\sqrt{17}, -\sqrt[3]{\pi})$.

1.2 CIRCLES AND THE SLOPE OF A LINE

1.2.1 Circles

The *circle* with center (h, k) and radius r consists of all points (x, y) whose distance from (h, k) is r. Using the formula for the distance from (x, y) to (h, k), you see that this circle consists of all points (x, y) such that

$$\sqrt{(x-h)^2+(y-k)^2} = r. \qquad (1)$$

Squaring both sides of (1), you obtain the equivalent relation

$$(x-h)^2+(y-k)^2 = r^2. \qquad (2)$$

Equation (2) is known as the *equation of the circle.*

Example 1 The equation of the circle with center $(-2, 4)$ and radius 5 is $(x-(-2))^2 + (y-4)^2 = 25$, or $(x+2)^2+(y-4)^2 = 25$. ‖

Example 2 The equation $(x+3)^2+(y+4)^2 = 18$ describes a circle with center at $(-3, -4)$ and radius $\sqrt{18} = 3\sqrt{2}$. ‖

Every equation of the form $ax^2 + ay^2 + bx + cy = d$ and satisfied by at least one point (x_1, y_1) is the equation of a circle. However, the general equation may have no locus in our real plane. For example, $x^2 + y^2 = -10$ has no real locus, for a sum of squares can't be negative. You should try to put any particular such equation in the form (2) to find the center and radius of the circle.

Example 3 Let us show that $3x^2 + 3y^2 + 6x - 12y = 60$ describes a circle.

SOLUTION We start by dividing by the common coefficient 3 of x^2 and y^2, and obtain

$$x^2 + y^2 + 2x - 4y = 20.$$

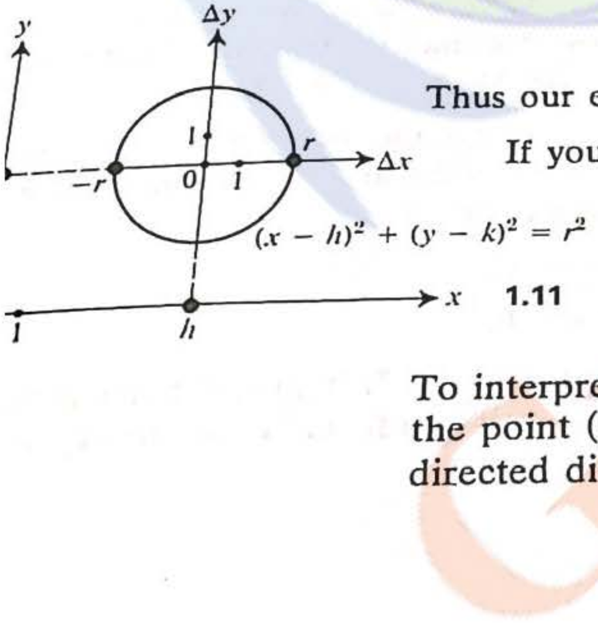
Completing the square Now we use the algebraic device of completing the square to get our equation in the form (2). The steps are as follows:

$$(x^2 + 2x) + (y^2 - 4y) = 20,$$
$$(x+1)^2 + (y-2)^2 = 20 + 1^2 + (-2)^2,$$
$$(x+1)^2 + (y-2)^2 = 25.$$

Thus our equation describes a circle with center $(-1, 2)$ and radius 5.

If you let $\Delta x = x - h$ and $\Delta y = y - k$, then (2) becomes

$$(\Delta x)^2 + (\Delta y)^2 = r^2.$$

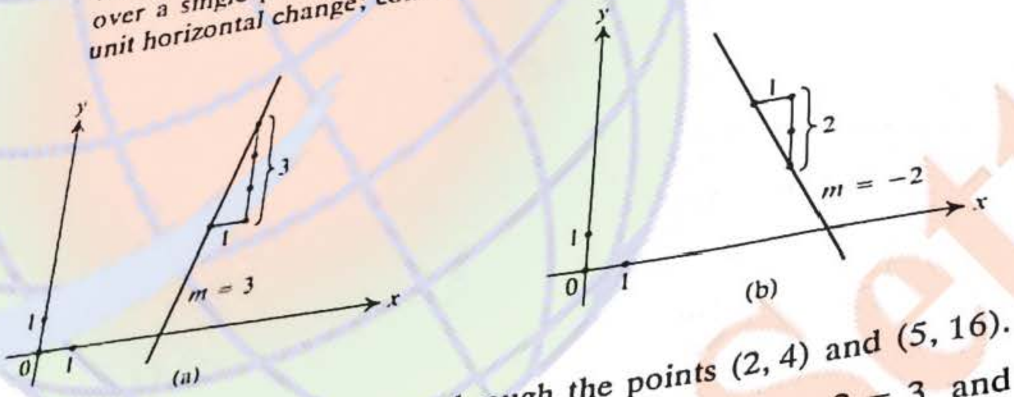


1.11

To interpret (3) geometrically, take a new Δx-axis and a new Δy-a
the point (h, k) as new origin, as shown in Fig. 1.11. Recall that Δ
directed distance from h to x and Δy is the directed distance fro

Translating axes Thus (3) is exactly the equation of the circle with respect to your new axes. This device is known as *translation of axes to* (h, k) and will often be useful. The equation $x^2 + y^2 = r^2$ describes a circle with center the origin of the x,y-coordinate system and radius r, while the equation $(\Delta x)^2 + (\Delta y)^2 = r^2$ describes a circle with center the origin of the $\Delta x, \Delta y$-coordinate system and radius r.

1.2.2 The slope of a line The **slope** m of a line is the number of units the line climbs (or falls) vertically for each unit of horizontal change from left to right. To illustrate, if a line climbs upward 3 units for each unit step you go to the right, as in Fig. 1.12(a), the line has slope 3. If a line falls 2 units downward per unit step to the right, as in Fig. 1.12(b), the line has slope -2. A horizontal line neither climbs nor falls, so it has slope 0. A vertical line climbs straight up over a single point, so it is impossible to measure how much it climbs per unit horizontal change; consequently *the slope of a vertical line is undefined.*



(a) (b)

1.12

Example 4 Let's find the slope of the line through the points (2, 4) and (5, 16).

SOLUTION As we go from (2, 4) to (5, 16), we have $\Delta x = 5 - 2 = 3$ and $\Delta y = 16 - 4 = 12$. Since the line climbed $\Delta y = 12$ units while we went $\Delta x = 3$ units to the right, and since a line climbs at a uniform rate, the amount it climbs per horizontal unit to the right is $\Delta y/\Delta x = \frac{12}{3} = 4$. ‖

As illustrated in Example 4, you can find the slope m of the line through (x_1, y_1) and (x_2, y_2) if $x_1 < x_2$ by finding Δx and Δy as you go from (x_1, y_1) to (x_2, y_2), and then taking the quotient; so

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{4}$$

We assume that our line is not vertical, so $x_1 \neq x_2$. If it should happen that $x_2 < x_1$, then, to go from left to right, you should go from (x_2, y_2) to (x_1, y_1), and you obtain

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}, \tag{5}$$

which is the same formula as in (4). In summary, the slope m of a nonvertical line through two points is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Difference of y-coordinates}}{\text{Difference of x-coordinates in the same order}}. \tag{6}$$

Example 5 The line through (7, 5) and (−2, 8) has slope

$$m = \frac{\Delta y}{\Delta x} = \frac{8 - 5}{-2 - 7} = \frac{3}{-9} = -\frac{1}{3}. \quad \|$$

Two lines are parallel precisely when they climb (or fall) at the same rate, that is, when they have equal slopes. Now suppose that two lines are perpendicular instead. Let one line have slope m_1 and the other have slope m_2. By translating axes, we may assume that our lines intersect at the origin. Then $(1, m_1)$ and $(1, m_2)$ are points on the lines, as shown in Fig. 1.13. The lines are perpendicular if and only if the triangle with vertices (0, 0), $(1, m_1)$, and $(1, m_2)$ satisfies the Pythagorean relation $d^2 = r^2 + s^2$.

From our distance formula, we obtain

$$r^2 = (1 - 0)^2 + (m_1 - 0)^2 = 1 + m_1^2,$$
$$s^2 = (1 - 0)^2 + (m_2 - 0)^2 = 1 + m_2^2,$$
$$d^2 = (1 - 1)^2 + (m_2 - m_1)^2 = (m_2 - m_1)^2.$$

The Pythagorean condition becomes

$$(m_2 - m_1)^2 = (1 + m_1^2) + (1 + m_2^2)$$

or

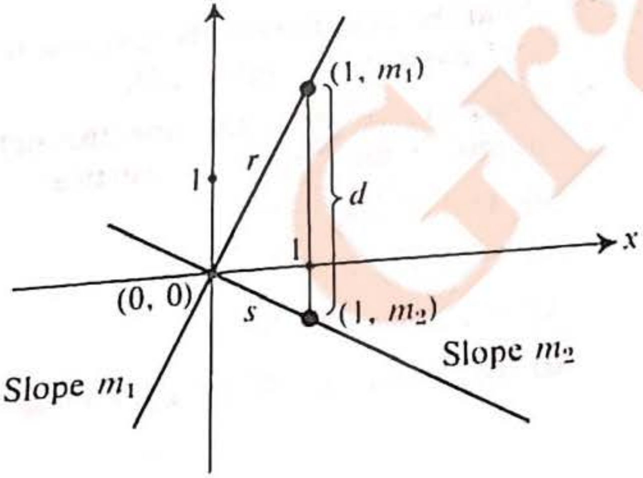
$$m_2^2 - 2m_1m_2 + m_1^2 = 2 + m_1^2 + m_2^2.$$

Therefore

$$-2m_1m_2 = 2;$$

so

$$m_1m_2 = -1 \quad \text{or} \quad m_2 = -\frac{1}{m_1}.$$



1.13

Example 6 Let's find the slope of a line perpendicular to the line through (6, −5) and (8, 3).

SOLUTION The given line has slope

$$\frac{\Delta y}{\Delta x} = \frac{3 - (-5)}{8 - 6} = \frac{8}{2} = 4;$$

so a perpendicular line has slope $-\frac{1}{4}$. ‖

SUMMARY

1. *The circle with center* (h, k) *and radius* r *has equation*

$$(x - h)^2 + (y - k)^2 = r^2.$$

2. *To find the center* (h, k) *and the radius* r *of a circle* $ax^2 + ay^2 + bx + cy = d$, *complete the square on the x-terms and on the y-terms.*
3. *A vertical line has undefined slope. If* $x_1 \neq x_2$, *the line through* (x_1, y_1) *and* (x_2, y_2) *has slope*

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

4. *Lines of slopes* m_1 *and* m_2 *are:*

parallel if and only if $m_1 = m_2$;
perpendicular if and only if $m_1 m_2 = -1$, *or* $m_2 = -1/m_1$.

EXERCISES

1. *Find the equation of the circle with the given center and radius.*
   - a) *center* (0, 0), *radius* 5
   - b) *center* (−1, 2), *radius* 3
   - c) *center* (3, −4), *radius* $\sqrt{30}$

. *Find the center and radius of the given circle.*

a) $(x-2)^2 + (y-3)^2 = 36$

b) $(x + 3)^2 + y^2 = 49$

) $(x + 1)^2 + (y + 4)^2 = 50$

nd the center and radius of the given circle.

$x^2 + y^2 - 4x + 6y = 3$

$^2 + y^2 + 8x = 9$

$^2 + 4y^2 - 12x - 24y = -\frac{9}{2}$

4. Find the equation of the circle with center in the second quadrant, tangent to the coordinate axes, and with radius 4.
5. Find the equation of the circle having the line segment with endpoints (−1, 2) and (5, −6) as a diameter.
6. Find the equation of the circle with center (2, −3) and passing through (5, 4).
7. Find the slope of the line through the indicated points, if the line is not vertical.
   - a) (−3, 4) and (2, 1)
   - b) (5, −2) and (−6, −3)
   - c) (3, 5) and (3, 8)
   - d) (0, 0) and (5, 4) e) (−7, 4) and (9, 4)

8. Find b so that the line through $(2, -3)$ and $(5, b)$ has slope -2.

9. Find a so that the line through $(a, -5)$ and $(3, 6)$ has slope 1.

10. Find the slope of a line perpendicular to the line through $(-3, 2)$ and $(4, 1)$.

11. Find b so that the line through $(8, 4)$ and $(4, -2)$ is parallel to the line through $(-1, 2)$ and $(2, b)$.

12. Show that the line joining the midpoints of two sides of a triangle is parallel to the third side. [*Hint.* Let the vertices of the triangle be $(0, 0)$, $(a, 0)$, and (b, c).]

13. Water freezes at 0° Celsius and 32° Fahrenheit, while it boils at 100° Celsius and 212° Fahrenheit. If points (C, F) are plotted in the plane, where F is the temperature in degrees Fahrenheit corresponding to a temperature of C degrees Celsius, then a straight line is obtained. Find the slope of the line. What does this slope represent in this situation?

14. The Easy Life Prefabricated Homes Company listed its super-deluxe ranch model for \$30,000 in 1960. The company increased the price by the same amount each year, and listed the same model for \$90,000 in 1980. Find the slope of the segment drawn through points (Y, C) in the plane, where Y could be any year from 1960 to 1980 and C is the cost of this model ranch house in that year. What does this slope represent in this situation?

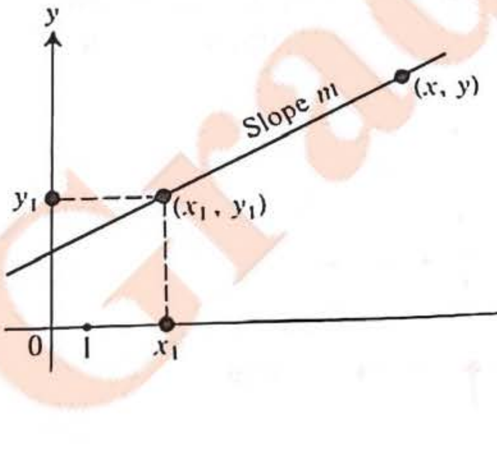
calculator exercises

15. Find the center and radius of the given circle.
    a) $(x - \pi)^2 + (y - \sqrt{\pi})^2 = 2.736$
    b) $x^2 + y^2 + 3.1576x - 1.2354y = 3.33867$
    c) $\sqrt{2}x^2 + \sqrt{2}y^2 - \pi^3 x + (\pi^2 + 3.4)y = \sqrt{17}$

16. Find the slope of the line through the indicated points.
    a) $(2.367, \pi)$ and $(\sqrt{3}, 8.9)$
    b) $(\pi^2, \sqrt[3]{19})$ and $(12.378, \sqrt{5.69})$
    c) $(\sqrt{2} + \sqrt{3}, \pi - \sqrt{19.3})$ and $(\sqrt{\pi} + 1.45, \sqrt{14} - \sqrt[5]{134})$

1.3 THE EQUATION OF A LINE

Let a given line have slope m and pass through the point (x_1, y_1) as shown in Fig. 1.14. Let's try to find an algebraic condition for a point (x, y) to lie on the line. If the slope of the line that joins (x_1, y_1) and (x, y) is also m, then that line is parallel to the given line, for they have the same slope. But bot



1.14

lines go through (x_1, y_1), so they must coincide. Therefore a condition for (x, y) to lie on the given line is that

$$\frac{y - y_1}{x - x_1} = m \tag{1}$$

or

$$y - y_1 = m(x - x_1). \tag{2}$$

Equation (2) is the *point–slope form* of the equation of the line.

Example 1 Let's find the equation of the line through $(2, -3)$ with slope 7.

SOLUTION The equation is $y - (-3) = 7(x - 2)$ or $y + 3 = 7(x - 2)$. This equation may be simplified to $y = 7x - 17$. The point $(3, 4)$ lies on this line, since $4 = 7 \cdot 3 - 17$. ‖

As indicated in Example 1, the point–slope equation (2) can be rewritten in the form

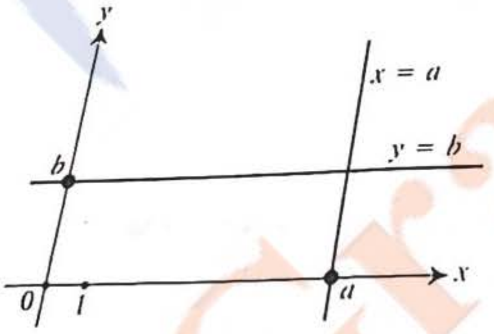
$$y = mx + b, \tag{3}$$

where $b = y_1 - mx_1$. The constant b in (3) has a nice interpretation. If you set $x = 0$ in (3), then $y = b$, so the point $(0, b)$ satisfies the equation and thus lies on the line. This point $(0, b)$ is on the y-axis, and b is the ***y*-intercept** of the line. For this reason, (3) is the *slope–intercept form* of the equation of the line. If the line crosses the x-axis at $(a, 0)$, then a is the ***x*-intercept** of the line.

Example 2 We find the intercepts of the line in Example 1.

SOLUTION The equation is $y = 7x - 17$, so -17 is the y-intercept. To find the x-intercept, you set $y = 0$ and obtain $7x - 17 = 0$, so $x = \frac{17}{7}$. Thus the point $(\frac{17}{7}, 0)$ lies on the line, so $\frac{17}{7}$ is the x-intercept. ‖

The vertical line through $(a, 0)$ in Fig. 1.15 has undefined slope, so it does not have an equation of the form (2) or (3). But surely a condition that (x, y) lie on the line is simply that $x = a$. Of course, $y = b$ is the horizontal line through $(0, b)$ shown in the figure. In any kind of coordinate system, it is important to know what loci are obtained by setting the coordinate variables equal to constants. We have seen that in our rectangular x,y-coordinate system, $x = a$ is a vertical line and $y = b$ is a horizontal line.



1.15

Any time you want to find the equation of a line, say to yourself, "I need to find a *point on the line* and the *slope of the line*." Then use Eq. (2).

Example 3 Let's find the equation of the line through $(-5, -3)$ and $(6, 1)$.

SOLUTION We solve the problem as follows.

POINT: $(x_1, y_1) = (-5, -3)$

SLOPE: $m = [1 - (-3)]/[6 - (-5)] = \frac{4}{11}$

EQUATION: $y + 3 = \frac{4}{11}(x + 5)$

The equation can be simplified to $11y + 33 = 4x + 20$ or $4x - 11y = 13$. ‖

Finally, observe that every equation $ax + by + c = 0$, where either $a \neq 0$ or $b \neq 0$ is the equation of a line. If $b = 0$, the equation becomes $x = -c/a$, which is a vertical line. If $b \neq 0$, the equation becomes $y = -(a/b)x - c/b$, which is a line with slope $m = -a/b$ and y-intercept $-c/b$.

SUMMARY

1. *A vertical line has equation* $x = a$.
2. *A horizontal line has equation* $y = b$.
3. *To find the equation of a line, find one point* (x_1, y_1) *on the line and the slope m of the line. The equation is then*

$$y - y_1 = m(x - x_1).$$

4. *The line* $y = mx + b$ *has slope m and y-intercept b.*

EXERCISES

1. Find the equation of the indicated line.
   a) Through $(-1, 4)$ with slope 5
   b) Through $(2, 5)$ and $(-3, 5)$
   c) Through $(4, -5)$ and $(-1, 1)$
   d) Through $(-3, 4)$ and $(-3, -1)$

Find the slope, x-intercept, and y-intercept of the indicated line.

a) $x - y = 7$
b) $y = 11$
c) $x = 4$
d) $7x - 13y = 8$

Find the equation of the line through $(-2, 1)$ and parallel to the line $2x + 3y = 7$.

Find the equation of the line through $(3, -4)$ erpendicular to the line $4x - 7y = 11$.

5. Are the lines $3x + 4y = 8$ and $4x + 3y = 14$ perpendicular? Why?
6. Are the lines $7x + 8y = 10$ and $8x - 7y = -14$ perpendicular? Why?
7. Find the equation of the perpendicular bisector o the line segment joining $(-1, 5)$ and $(3, 11)$.
8. Find the point of intersection of the lines $2x$ $3y = 7$ and $3x + 4y = -8$.
9. Find the distance from the point $(-2, 1)$ to line $3x + 4y = 8$.
10. Show that the perpendicular bisectors of the of a triangle meet at a point. [Hint. Le vertices of the triangle be $(-a, 0)$, $(a, 0$ (b, c).]

11. Find the equation of the circle through the points (1, 5), (2, 4), and (−2, 6).
12. Referring to Exercise 13 of the preceding section, find the linear relation giving the temperature F in degrees Fahrenheit corresponding to a temperature of C degrees Celsius.
13. A snowstorm starts at 3:00 A.M. and continues until 11:00 A.M. If there were 13 in. of old snow on the ground at the start of the storm and the new snow accumulates at a constant rate of $\frac{1}{2}$ in. per hour, find the depth d in inches at time of day t for $3 \leq t \leq 11$.

1.4 FUNCTIONS AND THEIR GRAPHS

1.4.1 Functions

The area enclosed by a circle is a *function* of the radius of the circle, meaning that the area depends on and varies with this radius. If a numerical value for the radius is given, the area enclosed by the circle is determined. For example, if the radius is 3 units, then the area is 9π square units. Similarly, the area of a rectangular region is a function of both the length and the width of the rectangle; that is, the area depends on and varies with these quantities. If the length of a rectangle is 5 units and the width is 3 units, the rectangle encloses a region that has an area of 15 square units.

The study of how one numerical quantity Q depends on and varies with other numerical quantities is one of the major concerns of science. A rule that specifies the numerical value of Q for all possible values of the other quantities is an exceedingly useful thing to have. Viewed intuitively, a *function* is such a rule.

In the next few chapters, we will be interested chiefly in the case where the value of a number y depends upon the value of some single number x, so that *y is a function of x*. This is often expressed by $y = f(x)$, and we consider *f to be* the function. We shall sometimes be sloppy and speak of the function $f(x)$, but strictly speaking, f is the function and $f(x)$ is the *value of the function f at x*. If you want to talk about several functions at once, use different letters. The letters f, g, and h are commonly used for functions.

For an example, perhaps $y = f(x) = \sqrt{x-1}$, so that

$$f(2) = \sqrt{2-1} = \sqrt{1} = 1,$$
$$f(5) = \sqrt{5-1} = \sqrt{4} = 2,$$

and

$$f(1) = \sqrt{1-1} = \sqrt{0} = 0.$$

Note that $f(0)$ is not defined for this function f, for we shall allow only real numbers, and $\sqrt{0-1} = \sqrt{-1}$ is not a real number. The set of those x-values allowed is the **domain** *of the function; and x, which may take on any number in that domain as value, is the* **independent variable**. *Similarly, y is the* **dependent variable**; *its value depends on the value of the variable x. The set of y-values obtained, as x goes through all values in the domain, is the* **range** *of the function.*

If $y = f(x)$, then f should assign to each x in the domain only one value y. This is a very important requirement. We just worked with the function f

given by $y = f(x) = \sqrt{x - 1}$, and said that $f(5) = \sqrt{5 - 1} = \sqrt{4} = 2$. You may have wondered why we didn't say $\sqrt{4} = \pm 2$. We want $\sqrt{x - 1}$ to define a *function*, and for this reason, we shall always use the $\sqrt{\ }$ symbol to mean the *nonnegative* square root. If we want the negative square root, we will always use $-\sqrt{\ }$.

When a function f is defined by a formula and the domain of the independent variable is not specifically given, we always consider the domain to consist of all values of x for which the formula can be evaluated and yields a *real* number. In particular, *division by zero is not allowed, and square roots (or fourth roots, or any even roots) of negative numbers are not allowed.*

Example 1 Let's find the domain of the function f given by the formula $y = f(x) = \sqrt{x - 1}$.

SOLUTION The domain consists of all x such that $x - 1 \geq 0$, or such that $x \geq 1$. The range of f consists of all $y \geq 0$. ‖

Example 2 At the start of this section we said that the area A of a circle is a function of its radius r. Let's describe this function.

SOLUTION If you call this function g, then r is the independent variable, A the dependent variable, and you have

$$A = g(r) = \pi r^2 \quad \text{for} \quad r \geq 0.$$

The domain constraint $r \geq 0$ must be stated because you can't have a circle of negative radius. Of course, you could compute πr^2 for negative values of r. This time, the domain restriction is due to the geometric origin of the function. ‖

Example 3 We find the domain of $y = f(x) = (x^2 - 1)/(x^2 - 9)$.

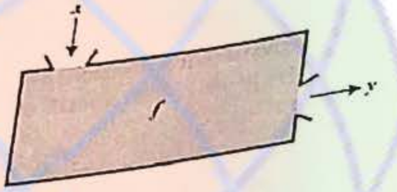
SOLUTION Note that $f(2) = 3/(-5)$ and $f(5) = \frac{24}{16} = \frac{3}{2}$, but $f(3)$ is not defined since *division by zero is not allowed.* The domain of the function consists of all $x \neq \pm 3$. ‖

Example 4 At the start of this section we said that the area A of a rectangular region is a function of its length ℓ and width w. If g is this function, it is customary to write

$$A = g(\ell, w) = \ell \cdot w \quad \text{for} \quad \ell \geq 0, w \geq 0.$$

This time there are *two* independent variables, ℓ and w. The domain restrictions $\ell \geq 0$ and $w \geq 0$ are again required because of the geometric origin of the function: A rectangle can't have negative length or width. ‖

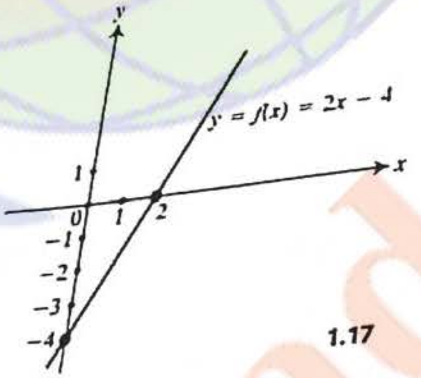
An intuitive picture of a function popular in elementary texts is a "black box," as in Fig. 1.16. One puts in a value of the independent variable x and a value of the dependent variable y comes out the other end. A "slide-rule" calculator is such a black (or other colored) box. For example, you punch in a value for x and then press the sin x button to "perform the function," and the y-value, where $y = f(x) = \sin x$, is shown as the display, usually to about eight-figure accuracy.



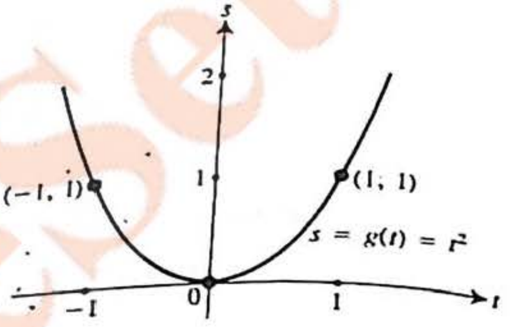
1.16

1.4.2 Graphs For a function f of one variable, we may find the points (x, y) in the plane where $y = f(x)$. These points form the **graph** of the function.

Example 5 Let $y = f(x) = 2x - 4$. The graph of this function is just the plane locus of the equation $y = 2x - 4$, which is the line with slope 2 and y-intercept -4 shown in Fig. 1.17. ‖



1.17



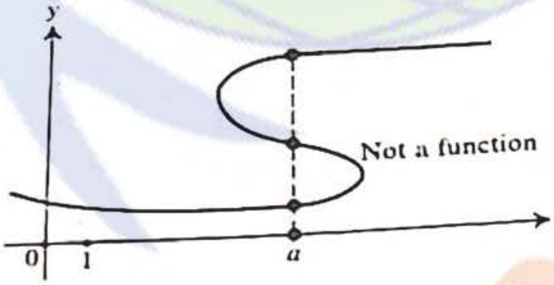
1.18

Example 6 The graph of the function $s = g(t) = t^2$ is shown in Fig. 1.18. Here we have used other letters for the variables and the function. ‖

As Examples 5 and 6 illustrate, the graph of a function given by an algebraic formula as $y = f(x)$ is the plane locus of points that satisfy this equation. The second paragraph of this section states that a function is a certain type of "rule." But what *is* a "rule"? Well, perhaps a "rule" is a "law." All right—now what is a "law"? We can keep this up for pages, and will still have to leave some term undefined. Mathematicians have recognized that at least one term *must* be left undefined, and have agreed that "set" shall be taken as an undefined term. So a mathematician who says that a function is a certain type of set has the professional right to refuse to answer if you ask what a set is. Now a function, $y = f(x)$, can be evaluated at any point x_1 in its domain if you know the point (x_1, y_1) on its graph. Thus the set of all points on the graph of a function can serve as a "rule" to evaluate a function. The collection of all such points can be viewed as a set. Here is a modern definition of a real-valued function of one real variable, but please don't get carried away or confused by this definition. Keep thinking of such a function as a rule, which is frequently given by some mathematical formula.

Definition 1.1 A **real-valued function** of one real variable is a set of ordered pairs (x, y) of real numbers such that no two different pairs have the same first coordinate.

The requirement of the definition that different pairs have different x-coordinates was illustrated in the discussion of $y = f(x) = \sqrt{x - 1}$ as a function. Recall that $f(5) = 2$, not ± 2. That is, $(5, 2)$ is one of the pairs of the function, but $(5, -2)$ is not. The curve in Fig. 1.19 is *not* the graph of a function $y = g(x)$, since *three* points have the same x-coordinate a.



1.19

One way to sketch the graph $y = f(x)$ is to make a table of corresponding values of x and y, plot the points, and draw a curve through them. Computing y-values can be tedious, and a calculator is often helpful. The calculator exercises of this section deal with such tables and plots. A computer can easily make such a table for many important functions. Printout 1.1 shows a table of x-values and y-values for the polynomial function f given by

$$y = f(x) = x^3 + 10x^2 + 8x - 50,$$

using 27 equally spaced x-values on the interval $[-10, 3]$. These results were obtained by using a computer program XYVALUES, written in the language BASIC and shown in Appendix 1. Tables of other functions can be found by changing just two lines (150 and 170) in the program XYVALUES.

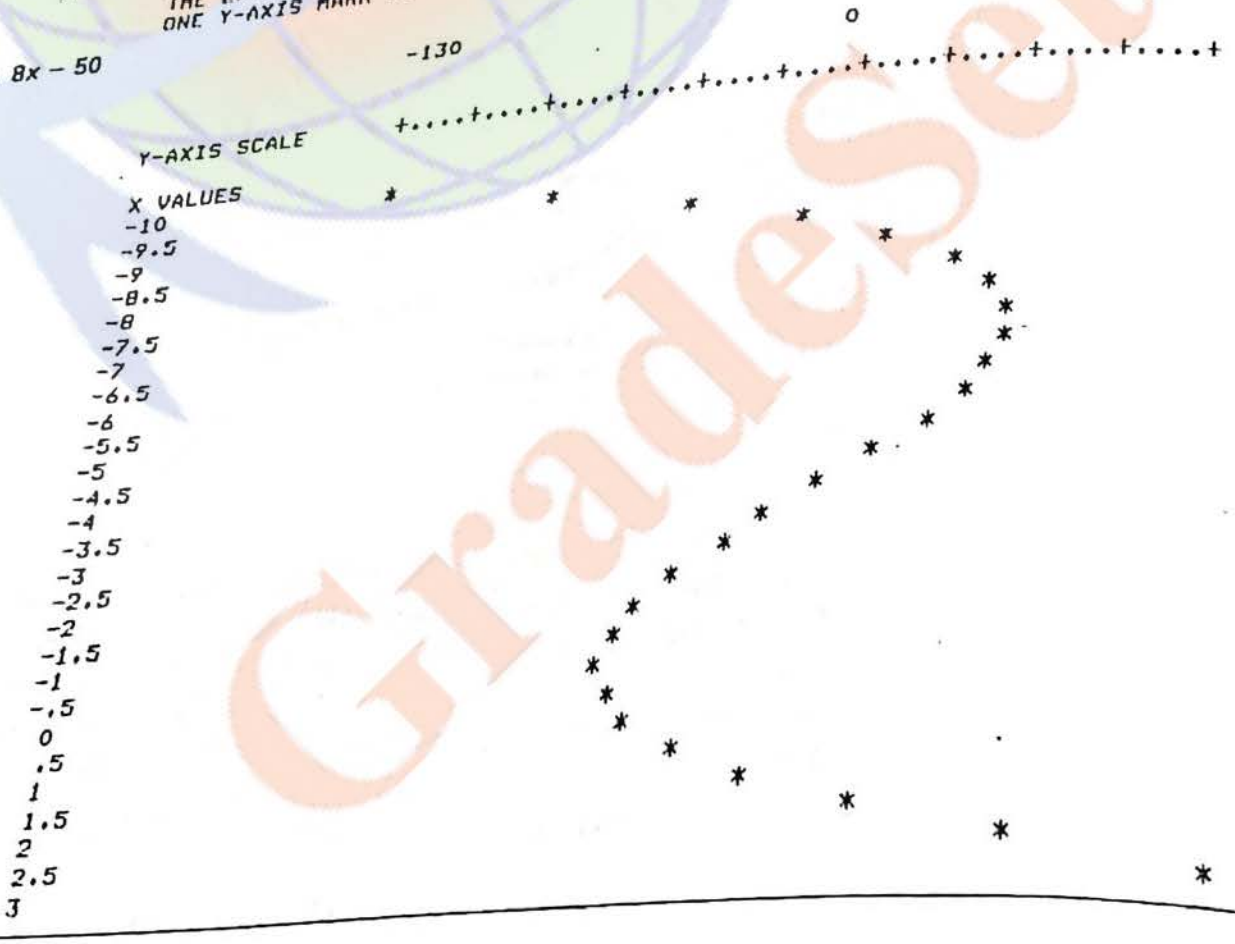
Plotting the graph from a table is still a bit of a nuisance. The computer can also give you a pretty good plot of the graph at a terminal. Printout 1.2 shows the graph for the data in Printout 1.1 as given by a program PLOT (see Appendix 1). In PLOT, the y-axis goes across the page and the x-axis down the page, so you have to rotate the page 90° counterclockwise to bring the axes into their usual position.

```
XYVALUES

X-VALUE        Y-VALUE
-10            -130
-9.5           -88.875
-9             -41
-8.5           -9.625
-8              14
-7.5            30.625
-7              41
-6.5            45.875
-6              46
-5.5            42.125
-5              35
-4.5            25.375
-4              14
-3.5            1.625
-3             -11
-2.5           -23.125
-2             -34
-1.5           -42.875
-1             -49
-.5            -51.625
0              -50
.5             -43.375
1              -31
1.5            -12.125
2               14
2.5             48.125
3               91
```

Printout 1.1
$y = x^3 + 10x^2 + 8x - 50$

Printout 1.2 $y = x^3 + 10x^2 + 8x - 50$

```
PLOT
INPUT ENDPOINTS A,B  OF INTERVAL OF X VALUES.
?-10,3
INPUT NUMBER <= 100 OF POINTS TO BE PLOTTED.
?27

THE SMALLEST Y VALUE IS -130
THE LARGEST Y VALUE IS  91
ONE Y-AXIS MARK EQUALS 4.42   UNITS
```



SUMMARY

1. *If $y = f(x)$, then x is the independent variable and y the dependent variable.*
2. *If $y = f(x)$, the domain of the function f consists of all allowable values of the variable x. The range of f consists of all values obtained for y as x goes through all values in the domain.*
3. *A function f assumes only one value $f(x)$ for each x in its domain. Thus $\pm\sqrt{x}$ is not a function.*
4. *If $y = f(x)$ is described by a formula, the domain of f consists of all x where $f(x)$ can be computed and gives a real number. For us, this usually means just excluding x-values that would lead to division by zero or to taking even roots of negative numbers.*
5. *The graph of f consists of all points (x, y) such that $y = f(x)$.*
6. *Graphs can be sketched by making a table of x- and y-values and plotting the points (x, y), although this may be hard work.*

EXERCISES

1. Express the volume V of a cube as a function of the length x of an edge of the cube.

2. Express the volume V of a cylinder as a function of the radius r of the cylinder and the length ℓ of the cylinder.

3. Express the area A enclosed by a circle as a function of the perimeter s of the circle.

4. Express the volume V of a box with square base as a function of the length x of an edge of the base and the area A of one side of the box.

5. Express the volume V of a cube as a function of the length d of a diagonal of the cube. (A diagonal of a cube joins a vertex to the *opposite vertex*, which is the vertex farthest away.)

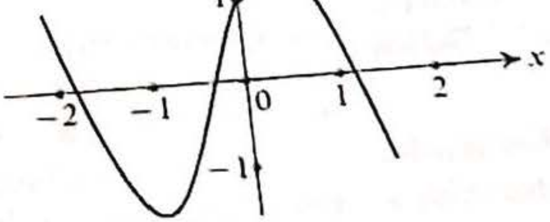
6. Bill starts at a point A at time $t = 0$ and walks in a straight line at a constant rate of 3 mi/hr toward point B. If the distance from A to B is 21 mi, express his distance s from B as a function of the time t, measured in hours.

7. Mary and Sue start from the same point on a level plain at time $t = 0$. Mary walks north at a constant rate of 3 mi/hr, while Sue jogs west at a constant rate of 5 mi/hr. Find the distance s between them as a function of the time t, measured in hours.

8. Smith, who is 6 ft tall, starts at time $t = 0$ directly under a light 30 ft above the ground, and walks away in a straight line at a constant rate o 4 ft/sec.
   a) Express the length ℓ of Smith's shadow as function of the distance x he has walked.
   b) Express the length ℓ of Smith's shadow a function of the time t, measured in second
   c) Express the distance x walked as a functio the length ℓ of his shadow.

9. A portion of the graph of a function f is show Fig. 1.20. Estimate each of the following fror graph.

   a) $f(0)$ b) $f(1)$ c) $f(-1)$



1.2

10. Let $f(x) = x^2 - 4x + 1$. Find the following.
    a) $f(0)$ b) $f(-1)$ c) $f(5)$
    d) A formula in terms of Δx for $f(2 + \Delta x)$

11. Let $g(t) = t/(1 - t)$. Find the following, if defined.
    a) $g(0)$ b) $g(1)$ c) $g(-1)$
    d) A formula in terms of Δt for
    $$\frac{g(2 + \Delta t) - g(2)}{\Delta t}$$

12. Find the domain of the function defined by the given algebraic expression.
    a) $f(x) = \frac{1}{x}$ b) $f(x) = \frac{1}{x^2 - 1}$
    c) $f(x) = \frac{x}{x^2 - 3x + 2}$ d) $g(t) = \sqrt{t + 3}$

13. Proceed as in Exercise 12.
    a) $f(u) = \sqrt{u^2 - 1}$ b) $g(t) = \frac{\sqrt{t - 2}}{t^2 - 16}$
    c) $h(x) = \sqrt{x - 4}$ d) $k(v) = \frac{v^2}{\sqrt{9 - v^2}}$

14. Sketch the graph of the following functions.
    a) $y = f(x) = x - 1$ b) $y = g(x) = -x^2$

15. Proceed as in Exercise 14.
    a) $s = g(t) = t^2 - 4$ b) $y = f(x) = \sqrt{1 - x^2}$
    c) $s = f(r) = -\sqrt{1 - r^2}$

16. Proceed as in Exercise 14.
    a) $y = f(x) = \frac{1}{x}$ b) $y = g(x) = \frac{1}{(x - 2)}$
    c) $y = h(u) = \frac{-1}{u}$

17. Make a table of x-values and y-values for the function
    $$y = f(x) = \frac{x + 1}{x - 1}$$
    for $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{5}{4}, \frac{3}{2}, 2, \frac{5}{2}$, and 3. Plot the points and draw the portion of the graph for all x in the domain such that $-1 \leq x \leq 3$.

calculator exercises

18. Make a table of values for the function $f(x) = (x + 1)/\sqrt{x^3 + 1}$ using 11 equally spaced x-values starting with $x = 0$ and ending with $x = 10$. Use the data to draw the graph of the function over $[0, 10]$. [*Note.* Eleven values give ten intervals.]

19. Make a table of values for the function $f(x) = \sin x^2$ using 13 equally spaced x-values starting with $x = 0$ and ending with $x = 3$. Use radian measure. Use the data to draw the graph of the function over $[0, 3]$. [*Note.* Thirteen values give twelve intervals. While we have not "defined" the function $\sin x^2$ yet, it is defined for you in your "black box" calculator.]

1.5 GRAPHS OF MONOMIAL AND QUADRATIC FUNCTIONS

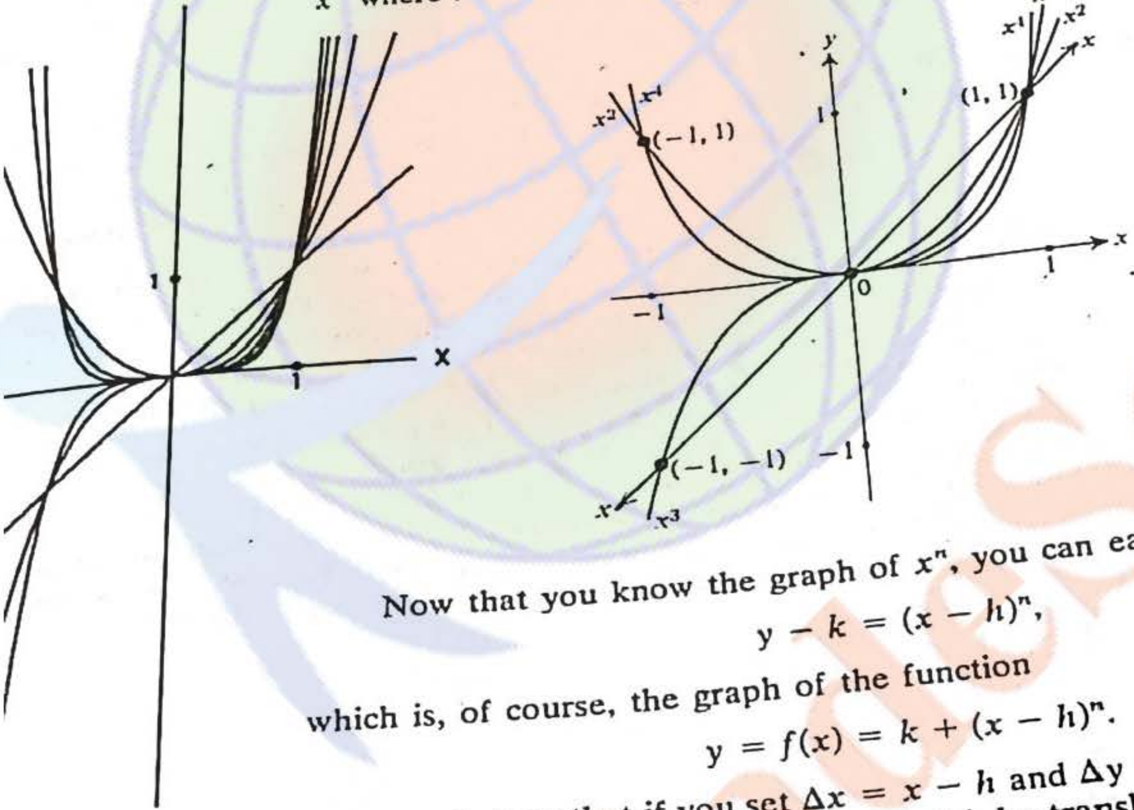
The monomial functions are those given by the monomials

$$x, \quad x^2, \quad x^3, \quad x^4, \quad x^5, \quad \ldots, \quad x^n, \quad \ldots$$

or constant multiples of them. When a function is given by a formula, we often refer to the formula as the function, to save writing. Thus we may refer to the function $4x^3$ rather than the function f where $f(x) = 4x^3$.

1.5.1 Monomial functions

It is important to know the graphs of the monomial functions. You know that the graph of the function x is a straight line of slope 1 through the origin. The graph of x^2 was shown in Fig. 1.18 in the preceding section. All the monomial graphs x^n go through the origin (0, 0) and the point (1, 1). If n is even, the graph of x^n goes through $(-1, 1)$ while, if n is odd, the graph goes through $(-1, -1)$. In Fig. 1.21 the graphs are indicated on one set of axes for easy comparison. Note in particular that the larger the value of n, the closer the graph is to the x-axis for $-1 < x < 1$. For example, $(\frac{1}{2})^4 < (\frac{1}{2})^2$, so the graph of x^4 is closer to the x-axis where $x = \frac{1}{2}$ than the graph of x^2. If $|x| > 1$, then the larger the value of n, the farther the graph of x^n is from the x-axis. For example, $2^5 > 2^3$, so x^5 is further from the x-axis than x^3 where $x = 2$.



1.21. Computer-generated figure (left); artist's figure (right).

Now that you know the graph of x^n, you can easily sketch

$$y - k = (x - h)^n, \tag{1}$$

which is, of course, the graph of the function

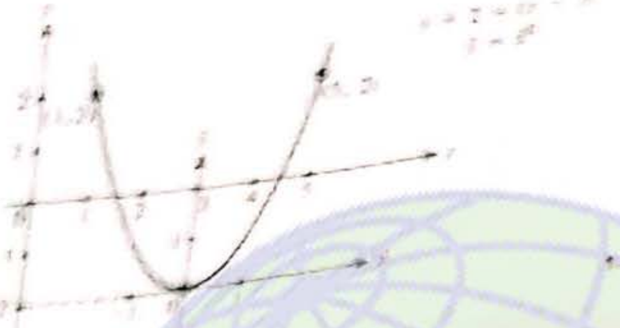
$$y = f(x) = k + (x - h)^n.$$

You saw before that if you set $\Delta x = x - h$ and $\Delta y = y - k$, so that Eq. (1) becomes $\Delta y = (\Delta x)^n$, then you can sketch by translating to new $\Delta x, \Delta y$-axes at (h, k). In this graph-sketching context, we shall drop the Δ-notation and use more conventional notation,

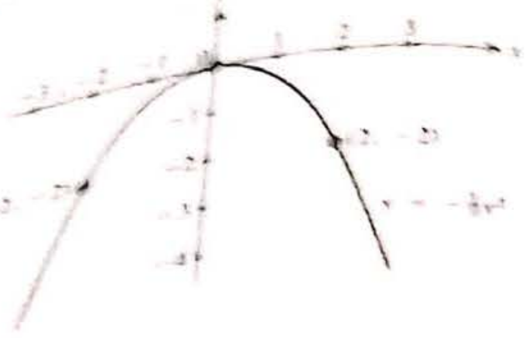
$$\bar{x} = x - h, \qquad \bar{y} = y - k;$$

Translating axes to (h, k)

so Eq. (1) becomes $\bar{y} = \bar{x}^n$. The graph of $y + 2 = (x - 3)^2$ is shown in Fig. 1.22. We translate to $\bar{x}, \bar{y}$-axes at the point $(3, -2)$, and our graph becomes $\bar{y} = \bar{x}^2$.



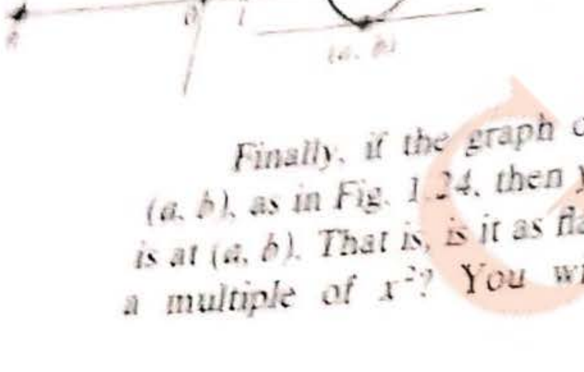
1.22



1.23

The graph of cx^n is much like the graph of x^n if $c > 0$. The points on cx^n are simply c times as far from the x-axis for each x-value as for x^n. Of course, if $c < 0$, then the graph is thrown to the other side of the x-axis. For example, the graph of $(-\frac{1}{2})x^2$ is shown in Fig. 1.23; it opens downward rather than upward.

Why are the graphs of monomial functions important? For the functions of most importance to us, we shall soon see that a very small piece of the graph near a point (h, k) on the graph looks a lot like a straight-line graph. Near that point, it can be approximated quite well by the line graph. We give an illustration of this in Fig. 1.24. Suppose the tangent line to the graph of f at (h, k) has slope m. Then, choosing $\bar{x},\bar{y}$-axes at (h, k), we find that the line $\bar{y} = m\bar{x}$ approximates the graph closely very near (h, k). In x,y-coordinates, the equation $\bar{y} = m\bar{x}$ becomes $y - k = m(x - h)$ or $y = k + m(x - h)$, which is the equation of the tangent line to the graph in x,y-coordinates. This problem of finding the tangent line is central to differential calculus, as we shall see in the following chapters.



1.24

Finally, if the graph of f has a horizontal tangent of slope $m = 0$ at (a, b), as in Fig. 1.24, then you would like to know just how "flat" the graph is at (a, b). That is, is it as flat as a multiple of x^4 is at $(0, 0)$, or only as flat as a multiple of x^2? You will see much later that, for many important functions, you can measure how "flat" the graph is at such a point by finding values of c and n such that $\bar{y} = c\bar{x}^n$ gives the best monomial approximation to the graph.

1.5.2 Quadratic functions

A quadratic function f is one of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. Graphs of these functions are called *parabolas*. By completing the square and translating to $\bar{x},\bar{y}$-axes, you can put the equation $y = ax^2 + bx + c$ in the form $\bar{y} = d\bar{x}^2$ for some constant d. That is, the graph of a quadratic function is just a translation of the graph of a quadratic monomial function. The reason for this becomes clear in following an example.

Example 1 Let's sketch the graph of the function $y = f(x) = -2x^2 - 6x - 2$.

SOLUTION Dividing by the coefficient -2 of x^2 and then completing the square, you obtain

Sketching a quadratic function

$$-\frac{y}{2} = x^2 + 3x + 1,$$

$$-\frac{y}{2} + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 + 1.$$

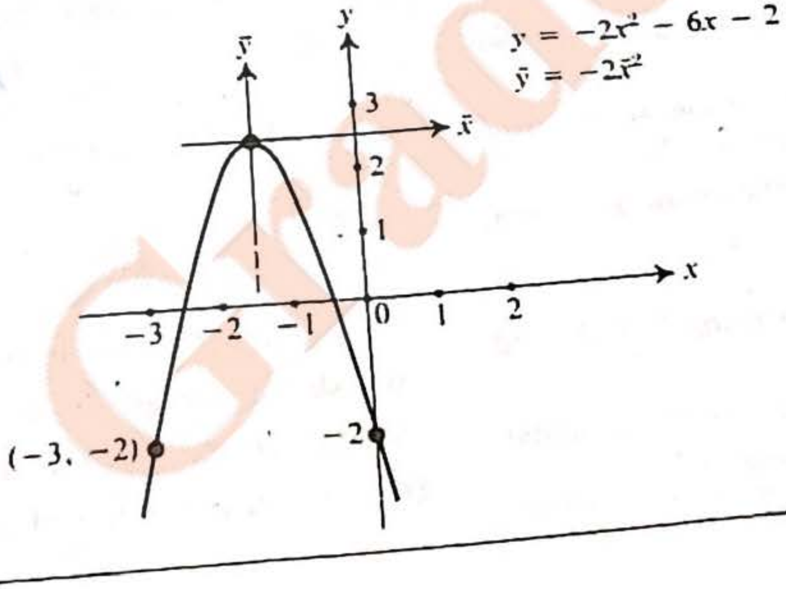
Now move all the constant terms to the lefthand side, obtaining

$$-\frac{y}{2} + \frac{9}{4} - 1 = \left(x + \frac{3}{2}\right)^2 \quad \text{or} \quad -\frac{y}{2} + \frac{5}{4} = \left(x + \frac{3}{2}\right)^2.$$

Finally, multiply back through by the -2,

$$y - \tfrac{5}{2} = -2(x + \tfrac{3}{2})^2.$$

Now set $\bar{x} = x + \frac{3}{2}$ and $\bar{y} = y - \frac{5}{2}$, which amounts to translating axes $(-\frac{3}{2}, \frac{5}{2})$. The equation becomes $\bar{y} = -2\bar{x}^2$. The graph is shown in F 1.25. ‖



1.25

SUMMARY

1. See Fig. 1.21 for the graphs of the monomial functions $x, x^2, x^3, \ldots$.
2. The graph of $y - k = (x - h)^n$ looks like the graph of $y = x^n$, but with the origin translated to (h, k).
3. Every quadratic equation $y = ax^2 + bx + c$, where $a \neq 0$, has a parabola as graph. The parabola can be sketched by completing the square on the x-terms and translating axes to put the equation in the form

$$\bar{y} = d\bar{x}^2.$$

EXERCISES

In Exercises 1 through 14, sketch the graph of the indicated function.

1. $-x^3$
2. $x^2 + 3$
3. $-x^4$
4. $x^2/2$
5. $-x^5/3$
6. $4 + (x - 2)^2$
7. $4x^3$
8. $(x + 1)^3 - 3$
9. $-1 - (x + 5)^4$
10. $x^2 + 2x + 1$
11. $x^2 - 4x + 3$
12. $-x^2 - 6x + 5$
13. $2x^2 - 4x + 6$
14. $-3x^2 + 6x - 12$

15. Sketch the graph of f where $f(x) = |x|$. Can you find one single line graph that approximates $f(x)$ well for a short distance on both sides at $x = 0$?

exercise sets for chapter 1

review exercise set 1.1

1. a) Find the directed length Δx from -2 to 5.
   b) Sketch all points (x, y) in the plane that satisfy $x > y + 1$.
2. a) Find the distance between $(2, -1)$ and $(-4, 7)$.
   b) Find the midpoint of the line segment joining $(-1, 3)$ and $(3, 9)$.
3. a) Find the equation of the circle with center $(2, -1)$ and passing through $(4, 6)$.
   b) Sketch all points (x, y) in the plane such that

$$(x - 1)^2 + (y + 2)^2 \leq 4.$$

) Find the slope of the line joining $(-1, 4)$ and $(3, 7)$.

Find the slope of a line that is perpendicular o the line through $(4, -2)$ and $(-5, -3)$.

5. a) Find the equation of the line through $(-4, 2)$ and $(-4, 5)$.
   b) Find the equation of the line through $(-1, 2)$ and parallel to the line $x - 3y = 7$.
6. a) Find the x-intercept and y-intercept of the line $3x + 4y = 12$.
   b) Find the point of intersection of the lines $x - 3y = 7$ and $2x - 5y = 4$.
7. Let

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 5x}.$$

a) Find the domain of f. b) Find $f(-2)$.

8. Sketch the graph of the function $f(x) = 1/x^2$.
9. Sketch the graph of the function $3 - (x + 4)^3$.
10. Sketch the graph of the function $2x^2 + 8x - 6$.

review exercise set 1.2

1. a) Sketch on the line all x such that $|x - 1| \leq 2$.
   b) Find the midpoint of the interval $[-5.3, 2.1]$.
2. a) Find the distance from $(-6, 3)$ to $(-1, -4)$.
   b) Sketch all points (x, y) in the plane such that $x \leq y$ and also $x \geq 1$.
3. a) Find the equation of the circle with $(-2, 4)$ and $(4, 6)$ as endpoints of a diameter.
   b) Find the center and radius of the circle $x^2 + y^2 - 6x + 8y = 11$.
4. Find c such that the line through $(-1, c)$ and $(4, -6)$ is perpendicular to the line through $(-2, 3)$ and $(4, 7)$.
5. a) Find the equation of the line through $(-1, 4)$ and $(3, 5)$.
   b) Find the equation of the vertical line through $(3, -7)$.
6. a) Find the equation of the line through $(-1, 3)$ with y-intercept 5.
   b) Find the equation of the line through $(2, 4)$ parallel to the line through $(0, 5)$ and $(2, -3)$.
7. Let $f(x) = \sqrt{25 - x^2}$.
   a) Find the domain of f.
   b) Find $f(3)$.
   c) Sketch the graph of f.
8. Express the distance from the origin to a point (x, y) on the line $2x - 3y = 7$ as a function of x only.
9. Sketch the graph of the function $2 + (x - 1)^4$.
10. Sketch the graph of the function $4 - 2x^2$.

more challenging exercises 1

1. Show that, for all real numbers a and b,
   a) $|a + b| \leq |a| + |b|$.
   b) $|a - b| \geq |a| - |b|$.
2. Prove that, for any numbers a_1, a_2, b_1, and b_2, you have

$$(a_1a_2 + b_1b_2)^2 \leq (a_1{}^2 + b_1{}^2)(a_2{}^2 + b_2{}^2).$$

3. Prove algebraically from Exercise 2 and the formula for distance that the distance from (x_1, y_1) to (x_3, y_3) is greater than or equal to the sum of the distance from (x_1, y_1) to (x_2, y_2) and the distance from (x_2, y_2) to (x_3, y_3). This is known as the *triangle inequality* in the plane. [*Hint*. Let $a_1 = x_2 - x_1$, $a_2 = x_3 - x_2$, $b_1 = y_2 - y_1$, and $b_2 = y_3 - y_2$, so that $x_3 - x_1 = a_2 - a_1$ and $y_3 - y_1 = b_2 - b_1$.]

. Show that if two circles

$$x^2 + y^2 + a_1x + b_1y = c_1$$

and

$$x^2 + y^2 + a_2x + b_2y = c_2$$

intersect in two points, the line through those points of intersection is

$$(a_2 - a_1)x + (b_2 - b_1)y = c_2 - c_1.$$

5. Find the distance from the point $(-3, 4)$ to the line

$$5x - 12y = 2.$$

6. Solve the inequality $x^2 + 4x < 1$ for x.
7. Find the equation of the smaller circle tangent t both coordinate axes and passing through th point $(-3, 6)$.
8. Find the distance between the lines

$$x - 2y = 15 \qquad \text{and} \qquad x - 2y = -3.$$

9. Find the minimum distance between the ci

$$x^2 + y^2 - 2x + 4y = 139$$

and

$$x^2 + y^2 + 4x - 6y = 3.$$

10. If $f(x) = (2x - 7)/(x + 3)$, find a function that $g(f(x)) = x$ for all x in the domain of